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Linear Algebra and its Applications 418 (2006) 215–224
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Symmetry transformations for square sliced three-way arrays, with applications to their typical rank

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Received 17 December 2004; accepted 6 February 2006
Available online 31 March 2006
Submitted by R.A. Brualdi

Abstract

The typical 3-tensorial rank has been much studied over algebraically closed fields, but very little has been achieved in the way of results pertaining to the real field. The present paper examines the typical 3-tensorial rank over the real field, when the slices of the array involved are square matrices. The typical rank of $3 \times 3 \times 3$ arrays is shown to be five. The typical rank of $p \times q \times q$ arrays is shown to be larger than $q + 1$ unless there are only two slices ($p = 2$), or there are three slices of order 2×2 ($p = 3$ and $q = 2$). The key result is that when the rank is q+1, there usually exists a rank-preserving transformation of the array to one with symmetric slices.

AMS classification: 15A69

Keywords: Three-way rank; Typical tensorial rank; Three-mode component analysis; Indscal; Candecomp; Parafac

0. Introduction

The rank of a three-way array (tensor) $\underline{\mathbf{X}}$ is defined as the smallest number of rank-one arrays that generate the array as their sum. The concept plays a role in Candecomp/Parafac decompositions [2,4], which decompose the p slices, of order $q \times r$, of the array in R components as

$$\mathbf{X}_i = \mathbf{A}\mathbf{C}_i\mathbf{B}', \tag{1}$$

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2006.02.002

where $\mathbf{A}$ is a $q \times R$ matrix, $\mathbf{B}$ is an $r \times R$ matrix and $\mathbf{C}_i$ $(R \times R)$ is diagonal, $i = 1, \ldots, p$. The smallest value of R which allows the decomposition (1) is equal to the rank of $\underline{\mathbf{X}}$ [5,7]. The typical rank of an array format refers to the rank arrays of that format have with positive probability.

The typical tensorial rank of three-way arrays over algebraically closed fields has been much studied in the context of computational complexity theory. For instance, Bürgisser et al. [1] gives a number of results for various classes of arrays. The study of tensorial rank over the real field has lagged behind. Ten Berge [13] gives results for "tall" $p \times q \times r$ arrays where $qr - q < p \leqslant qr$. The typical rank is also known when $qr < p$ (very tall arrays). For arrays where p, q, and r are closer to each other, hardly anything is known. The present paper narrows down the possible values of typical rank for such array formats. Specifically, we examine the typical rank of arrays with square slices, that is, with $q = r$. When $p = 2$, square slice arrays have typical rank $\{q, q + 1\}$ [12]. In the present paper, we focus on square cases with $p > 2$. We derive bounds for the typical rank of such arrays, and prove the typical rank of the "cubic" $3 \times 3 \times 3$ array to be 5.

Some typical ranks of square arrays can be found in [13]. Because adding slices cannot reduce the rank, just like removing slices cannot increase rank, we can also derive bounds to the typical rank from known values. For arrays of up to 8 slices, of order 5×5 or less, Table 1 summarizes what is known about typical rank. The notation $\{R, R + 1\}$ indicates that rank R and rank $R + 1$, but no other ranks, arise with positive probability, whereas $i \leqslant R \leqslant j$ means that ranks less than i and larger than j have probability zero.

All cells involving inequalities are simply based on the fact that adding slices cannot reduce rank. Clearly, for cases where p is equal to or close to q, only bounds are available, except when $p = 2$, or when $p = 3$ and $q = 2$. The purpose of this paper is to give typical ranks for some of the other cases, or at least narrow down the typical ranks by offering sharper bounds.

Because a $2 \times q \times q$ array has typical rank $\{q, q + 1\}$, the typical rank of a $p \times q \times q$ array when $p > 2$ is at least q. Ten Berge and Smilde [17] have shown that, when $p > 2$, the $p \times q \times q$ array has rank q with probability zero. Their result allows us to discard q as a rank value of positive volume for $p \times q \times q$ arrays with $p > 2$. The main result of the present paper allows us, for $p \times q \times q$ arrays with $p > 2$, to also discard $q + 1$ as a value of positive probability, except when $p = 3$ and $q = 2$. This will be done by introducing a symmetry criterion.

1. A symmetry criterion for rank $q + 1$

A key question about square arrays is whether or not the square slices of the array can be transformed to symmetry.

Table 1
Typical rank values R for $p \times q \times q$ arrays

	$q = 2$	$q = 3$	$q = 4$	$q = 5$
$p = 2$	{2, 3}	{3, 4}	{4, 5}	{5, 6}
$p = 3$	3	$3 \leqslant R \leqslant 6$	$4 \leqslant R$	$5 \leqslant R$
$p = 4$	4	$4 \leqslant R \leqslant 6$	$4 \leqslant R$	$5 \leqslant R$
$p = 5$	4	$4 \leqslant R \leqslant 6$	$4 \leqslant R$	$5 \leqslant R$
$p = 6$	4	6	$6 \leqslant R$	$6 \leqslant R$
$p = 7$	4	7	$7 \leqslant R$	$7 \leqslant R$
$p = 8$	4	8	$8 \leqslant R$	$8 \leqslant R$

Result 1. *When a $p \times q \times q$ array (with slices $\mathbf{X}_1, \ldots, \mathbf{X}_p$), randomly sampled from a pqq-dimensional continuous distribution, has rank $q+1$, it is almost surely possible to find nonsingular matrices* $\mathbf{S}$ *and* $\mathbf{T}$ *such that* $\mathbf{SX}_i\mathbf{T}$ *is symmetric*, $i = 1, \ldots, p$.

Proof. Let the $p \times q \times q$ array have rank q+1. Then, because rank($\mathbf{X}_i$) is q almost surely, we can write $\mathbf{X}_i = \mathbf{AC}_i\mathbf{B}'$, $i = 1, \ldots, p$, for certain $q \times (q+1)$ matrices $\mathbf{A}$ and $\mathbf{B}$ of rank q almost surely, and a diagonal matrix $\mathbf{C}_i$. Premultiply $\mathbf{A}$ and the slices $\mathbf{X}_i$ by the inverse of the matrix containing the first q columns of $\mathbf{A}$. Also, premultiply $\mathbf{B}$ and the transposed slices $\mathbf{X}_i'$ by the inverse of the matrix containing the first q columns of $\mathbf{B}$. For instance, when $q = 3$ we obtain transformed versions $\mathbf{A}^*$ and $\mathbf{B}^*$ of $\mathbf{A}$ and $\mathbf{B}$, respectively, of the form

$$\mathbf{A}^* = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_3 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^* = \begin{bmatrix} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{bmatrix} \tag{2}$$

for certain nonzero scalars in the fourth columns. Next, rescale the rows of $\mathbf{A}^*$ and $\mathbf{B}^*$ such that the last columns have all elements 1, and apply the same transformations to the slices. When $q = 3$, this yields

$$\mathbf{A}^+ = \begin{bmatrix} a_1^{-1} & 0 & 0 & 1 \\ 0 & a_2^{-1} & 0 & 1 \\ 0 & 0 & a_3^{-1} & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^+ = \begin{bmatrix} b_1^{-1} & 0 & 0 & 1 \\ 0 & b_2^{-1} & 0 & 1 \\ 0 & 0 & b_3^{-1} & 1 \end{bmatrix}. \tag{3}$$

Because all corresponding columns of $\mathbf{A}^+$ and $\mathbf{B}^+$ are proportional, the transformations have resulted in new slices $\mathbf{A}^+\mathbf{C}_i\mathbf{B}^{+\prime}$ which are symmetric. □

Having found a necessary condition for rank $(q+1)$, it becomes important to determine when this condition can be satisfied.

Result 2. *When* $p = 2$, *or when* $p = 3$ *and* $q = 2$, *the transformation to symmetry, described in Result* 1, *is possible almost surely. In all other cases, it is impossible almost surely.*

Proof. Suppose the transformation to symmetry is possible. Then there exist nonsingular matrices $\mathbf{S}$ and $\mathbf{T}$ such that $\mathbf{SX}_i\mathbf{T}$ is symmetric. Hence, $(\mathbf{T}')^{-1}\mathbf{SX}_i\mathbf{TT}^{-1} = (\mathbf{T}'^{-1}\mathbf{S})\mathbf{X}_i$ is also symmetric. It follows that there exists a nonsingular matrix $\mathbf{W} = (\mathbf{T}')^{-1}\mathbf{S}$ such that $\mathbf{WX}_i$ is symmetric. Define $\mathbf{w} = [\mathbf{w}_1'|\mathbf{w}_2'|\cdots|\mathbf{w}_q']'$ as the column vector containing the rows of $\mathbf{W}$. Let $\mathbf{X}_i$ have columns $\mathbf{x}_{i1}, \ldots, \mathbf{x}_{iq}$. Symmetry of $\mathbf{WX}_i$ is equivalent to $\mathbf{w}_j'\mathbf{x}_{ik} = \mathbf{w}_k'\mathbf{x}_{ij}$ for all $j \neq k$. Hence, if $\mathbf{WX}_i$ is symmetric, $\mathbf{w}$ is orthogonal to the columns of a certain $q^2 \times q(q-1)/2$ matrix $\mathbf{H}_i$. Each column of $\mathbf{H}_i$ corresponds to one pair $j \neq k$ and contains q subvectors of order q, such that $-\mathbf{x}_{ik}$ is the jth subvector, $\mathbf{x}_{ij}$ is the kth subvector, and the remaining $q-2$ subvectors are zero. For example, when $q = 4$, we have $\mathbf{H}_i$ of the form

$$\mathbf{H}_i = \begin{bmatrix} -\mathbf{x}_{i2} & -\mathbf{x}_{i3} & -\mathbf{x}_{i4} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{x}_{i1} & \mathbf{0} & \mathbf{0} & -\mathbf{x}_{i3} & -\mathbf{x}_{i4} & \mathbf{0} \\ \mathbf{0} & \mathbf{x}_{i1} & \mathbf{0} & \mathbf{x}_{i2} & \mathbf{0} & -\mathbf{x}_{i4} \\ \mathbf{0} & \mathbf{0} & \mathbf{x}_{i1} & \mathbf{0} & \mathbf{x}_{i2} & \mathbf{x}_{i3} \end{bmatrix}, \tag{4}$$

where $\mathbf{0}$ is a vector of 4 zeros. Collect the p matrices $\mathbf{H}_i$ in a $q^2 \times pq(q-1)/2$ matrix $\mathbf{H} = [\mathbf{H}_1|\cdots|\mathbf{H}_p]$. Symmetry of $\mathbf{WX}_i$, $i = 1, \ldots, p$, implies that the vector $\mathbf{w}$ is orthogonal to all $pq(q-1)/2$ columns of $\mathbf{H}$. This is possible for a nonzero $\mathbf{w}$ if and only if the rank of $\mathbf{H}$ is less than q^2. It can be verified that the rank of $\mathbf{H}_i$ is almost surely $q(q-1)/2$. From the fact that $\mathbf{X}_1,\ldots,\mathbf{X}_p$

are randomly sampled from a continuous distribution, and the form of $\mathbf{H}_i$ in (4), it follows that the rank of $\mathbf{H}$ is almost surely min $\{q^2, pq(q-1)/2\}$, see Appendix A for more details. Hence, the equation $\mathbf{w}'\mathbf{H} = \mathbf{0}'$ almost surely cannot have a nonzero solution unless $q^2 > pq(q-1)/2$, or equivalently, unless $p + 2q - pq > 0$. This condition is satisfied when $p = 2$ and q arbitrary, and also when $p = 3$ and $q = 2$, but in no other case. The vector $\mathbf{w}$ can be chosen to imply a matrix $\mathbf{W}$ that will be nonsingular almost surely. □

It has been assumed that $\mathbf{X}_1, \ldots, \mathbf{X}_p$ are randomly sampled asymmetric matrices. However, when $\mathbf{X}_1, \ldots, \mathbf{X}_p$ are symmetric (as a result of a different sampling scheme), $\mathbf{H}$ will have linearly dependent rows, whence $\mathbf{w}'\mathbf{H} = \mathbf{0}'$ can always be solved for by a nonzero $\mathbf{w}$. Clearly, $\mathbf{w} = \mathrm{Vec}(\mathbf{I}_q)$ is always among the possible solutions, if not the only possible solution.

Corollary 1. *A $p \times q \times q$ array has rank $q + 1$ with probability zero, unless either $p = 2$ and q arbitrary, or $p = 3$ and $q = 2$.*

Proof. When the rank is $q + 1$, there will almost surely be a transformation to symmetry by virtue of Result 1. This transformation does not exist almost surely, except when $p = 2$, or when $p = 3$ and $q = 2$, see Result 2. □

In retrospect, Table 1 reflects the very cases where rank q+1 may have positive probability by Corollary 1. In addition, however, the corollary allows us to tighten the lower bounds in Table 1. When $p > 2$ and $q > 2$, all typical ranks are at least $q + 2$. Combining Result 2 again with the general property that larger format arrays (higher q for fixed p or higher p for fixed q) cannot have lower typical ranks than smaller format arrays, we obtain the improved bounds of Table 2.

For two specific cases, bounds can be replaced by exact values. First, for the case $p = q = 3$, Kruskal [8] and Rocci [10] have proven that the maximal rank is 5. From Corollary 1, it is now clear that rank less than 5 has probability zero. Therefore, the $3 \times 3 \times 3$ array has typical rank 5. Kruskal [6] conjectured that rank 4 might also arise with positive probability. It is now clear that this is not the case. It may be noted, however, that Corollary 1 relies on full random sampling of the slices. It does not apply when the array is forced to have symmetric slices. The typical rank is indeed 4 in case of symmetric slices [16].

The other case where exact values can be given is that of $p = 5$ and $q = 3$, where the typical rank is {5, 6} [14].

It may be noted that the lower bounds derived from Corollary 1, and recorded in Table 2 are sharp in at least one case. Specifically, $3 \times 4 \times 4$ arrays have rank 6 with positive probability. A proof is given in Appendix B.

Table 2
Improved typical rank values for $p \times q \times q$ arrays

	$q = 2$	$q = 3$	$q = 4$	$q = 5$
$p = 2$	{2, 3}	{3, 4}	{4, 5}	{5, 6}
$p = 3$	3	5	$6 \leqslant R$	$7 \leqslant R$
$p = 4$	4	$5 \leqslant R \leqslant 6$	$6 \leqslant R$	$7 \leqslant R$
$p = 5$	4	{5, 6}	$6 \leqslant R$	$7 \leqslant R$
$p = 6$	4	6	$6 \leqslant R$	$7 \leqslant R$
$p = 7$	4	7	$7 \leqslant R$	$7 \leqslant R$
$p = 8$	4	8	$8 \leqslant R$	$8 \leqslant R$

2. Incomplete transformation to symmetry

Result 2 and Corollary 1 can be further extended. We have shown that, when $p + 2q - pq > 0$, a transformation to full symmetry is possible. Whenever that arises, we know that the typical rank of a square array coincides with that of a symmetric array of the same format. Therefore, we may resort to Ten Berge et al. [16], where typical rank for values have been derived for cases where the square slices are sampled to be symmetric. This has not disclosed any typical rank values we did not know already, because the very cases where the transformation to symmetry was possible had already been solved. However, apart from attaining symmetry by a (rank-preserving) transformation, we may also instill symmetry by subtracting rank-one arrays. For instance, in a $p \times 2 \times 2$ array, the p slices can be made symmetric by subtracting a rank-one array with slices proportional to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Obviously, subtracting the rank-one array is not rank-preserving: It costs one additional component, which has to be taken into account. For instance, the fact that a square $p \times 2 \times 2$ array can be rendered symmetric by "paying" one dimension implies that the typical rank of such arrays is at most one higher than that of the symmetric $p \times 2 \times 2$ array. Specifically, square $p \times 2 \times 2$ arrays have typical rank 4 when $p > 3$, which is precisely one higher than that of the symmetric counterpart [16]. Although this does not offer new results, it does clarify some of the differences between typical ranks for symmetric and nonsymmetric arrays.

A more general approach is to combine an *incomplete* transformation to symmetry with subtracting a rank-one array. An example is the $3 \times 3 \times 3$ array. The matrix $\mathbf{H}_i$ of (4) is of order 9×3, and $\mathbf{H}$ is a 9×9 matrix. When we remove, say, the first column of each $\mathbf{H}_i$, and find $\mathbf{w}$ orthogonal to the remaining 9×6 matrix, we get a partial transformation to symmetry, with only the elements $(1, 2)$ differing from the elements $(2, 1)$ in each slice. By subtracting a rank-one array, with slices proportional to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we can attain full symmetry. Therefore, the typical rank of a square $3 \times 3 \times 3$ array is at most one higher than that of a symmetric $3 \times 3 \times 3$ array. As has been shown above, the former is 5, and the latter is 4 [16]. In general, we have

Result 3. *When $p < \frac{2q^2}{q^2-q-2}$, the typical rank of a square array is at most one higher than that of its symmetric counterpart.*

Proof. Suppose we omit column h from each $\mathbf{H}_i$, leaving asymmetry in only one pair of off-diagonal elements of the slices. Then, because p columns are deleted, the reduced version $\mathbf{H}_r$ of $\mathbf{H}$ is of order $q^2 \times \{pq(q-1)/2 - p\}$. Hence, $\mathbf{H}_r$ is of rank less than q^2 when $q^2 > pq(q-1)/2 - p$, which is the condition of this result. Upon constructing a matrix $\mathbf{W}$ from a vector $\mathbf{w}$ orthogonal to the columns of $\mathbf{H}_r$, a transformation to partial symmetry is obtained, after which full symmetry is attained by subtracting a rank-one array. □

Result 3 relates typical rank values of square arrays to those of their symmetric counterparts. Cases of Table 2 where the condition of Result 3 is satisfied are the $3 \times 3 \times 3$ array (already discussed), the $3 \times 4 \times 4$ array, and the $4 \times 3 \times 3$. For the $4 \times 3 \times 3$ array, the typical rank in case of symmetry is known to be $\{4, 5\}$. It follows that, for square arrays of that format, ranks above 6 do not occur with positive probability. Again, this fact had already been established by other means. Result 3 serves to clarify this.

Further extensions can be obtained when transformations to incomplete symmetry are combined with subtracting *two* rank-one arrays. This yields symmetry when $p < \frac{2q^2}{q^2-q-4}$. For instance, when $p = 8$ and $q = 3$, we have typical rank 6 for symmetric arrays [16], implying that the typical rank is at most 8 otherwise. Again, the typical rank has already been proven to be 8 in this case. Result 3 and further generalizations serve to explain known differences between typical rank for symmetric and square arrays.

3. Applications

The typical rank of a three-way array is the smallest number of components sufficient for a perfectly fitting Candecomp/Parafac decomposition. This can be used as a baseline for incomplete decompositions, to indicate what constitutes a "relatively small" number of components. This is similar to Principal Component Analysis, where the rank of a correlation matrix indicates the maximum number of components needed for a perfect fit.

Recently, typical rank has also found applications in Tucker three-way component analysis [18], where a three-way array is decomposed with the help of a so-called core array. Often, such core arrays are constrained to have a large majority of zero elements. However, it is well-known [15,11] that rank-preserving transformations of the core may produce a vast majority of zero elements. Hence, we need general rules to distinguish artifactual simplicity, to be attained by rank-preserving transformations, from nontrivial patterns of zeros, which truly represent statistical models. Such rules might be derived from maximal simplicity results [11], but typical rank results also apply. That is, when a simple core is hypothesized with rank less than typical rank, that core could not be obtained from any arbitrary core by rank-preserving transformations. Ten Berge and Smilde [17] have given an example of this in a constrained Tucker three-way component analysis context. They used a lower bound 6 to the typical rank of a $3 \times 5 \times 5$ array to show that a hypothesized core array with only 5 nonzero elements, and therefore of rank 5 or less, could not be attained by rank-preserving transformations. Incidentally, the present paper has improved this lower bound to 7.

4. Discussion

This paper has offered a partial explanation for the phenomenon that the typical rank of an array with square slices sometimes coincides with that of an array of the same format with symmetric slices. However, we have by no means explained all such cases. For instance, a $3 \times 3 \times 6$ array has typical rank 6, regardless of symmetry of the slices. An explanation for cases like this remains an open problem.

Appendix A. The matrix H has full rank with probability 1

First, we consider the case $p \geqslant q$. We define the $q \times p$ matrices $\mathbf{U}_k = [\mathbf{x}_{1k}|\cdots|\mathbf{x}_{pk}]$ for $k = 1, \ldots, q$. Hence, the matrix $\mathbf{U}_k$ contains the kth columns of slices $\mathbf{X}_1, \ldots, \mathbf{X}_p$. The matrix $\mathbf{H}$ has order $q^2 \times pq(q-1)/2$. For $q = 2$, the order of $\mathbf{H}$ is $q^2 \times p$ and we have $\mathbf{H} = \begin{bmatrix} -\mathbf{U}_2 \\ \mathbf{U}_1 \end{bmatrix}$. Since the elements of $\mathbf{H}$ are randomly sampled from a pq^2-dimensional continuous distribution, it follows that $\mathbf{H}$ has full rank with probability 1. For $p \geqslant q \geqslant 3$, $\mathbf{H}$ is either square (for $p = q = 3$) or horizontal (in all other cases). After a column permutation, $\mathbf{H}$ contains a $q^2 \times q^2$ submatrix $\mathbf{G}$ of the form

$$\mathbf{G} = \left[\begin{array}{c|cccccc} \mathbf{O} & -\mathbf{V}_2 & -\mathbf{V}_3 & \cdots & \cdots & -\mathbf{V}_q \\ \hline -\mathbf{V}_3 & \mathbf{V}_1 & \mathbf{O} & \cdots & \cdots & \mathbf{O} \\ \mathbf{V}_2 & \mathbf{O} & \mathbf{V}_1 & \ddots & & \vdots \\ \mathbf{O} & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \cdots & \cdots & \mathbf{O} & \mathbf{V}_1 \end{array}\right], \tag{5}$$

where $\mathbf{V}_k$ is the $q \times q$ matrix containing the first q columns of $\mathbf{U}_k$. Clearly, $\mathbf{H}$ has full rank if $\det(\mathbf{G}) \neq 0$. Since $\det(\mathbf{G})$ is an analytic function of the elements of $\mathbf{V}_k$, $k = 1, \ldots, q$, it follows from Fisher [3, Theorem 5.A.2] that $\det(\mathbf{G}) \neq 0$ with probability 1 if $\det(\mathbf{G}) \neq 0$ for one set of $\mathbf{V}_k$, $k = 1, \ldots, q$. Indeed, such a set of $\mathbf{V}_k$ exists. We take $\mathbf{V}_1 = \mathbf{I}_q$, which yields $\det(\mathbf{G}) = \det(\mathbf{V}_3\mathbf{V}_2 - \mathbf{V}_2\mathbf{V}_3)$; e.g., see [9, Section 2.11]. If q is even, we may take $\mathbf{V}_2$ as block diagonal with blocks $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{V}_3$ as block diagonal with blocks $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $(\mathbf{V}_3\mathbf{V}_2 - \mathbf{V}_2\mathbf{V}_3)$ is block diagonal with blocks $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and, hence, $\det(\mathbf{G}) = \pm 1$. If q is odd, we may take $\mathbf{V}_2$ as block diagonal with blocks $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and one final block $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}$; and we may take $\mathbf{V}_3$ as block diagonal with blocks $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and one final block $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Then the final 3×3 block of $(\mathbf{V}_3\mathbf{V}_2 - \mathbf{V}_2\mathbf{V}_3)$ becomes $\begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $\det(\mathbf{G}) = \pm 2$. Hence, we have shown that $\det(\mathbf{G}) \neq 0$ with probability 1. Therefore, $\mathbf{H}$ has full rank with probability 1 if $p \geqslant q \geqslant 3$.

Next, we consider the case $p < q$. When $p = 2$, $\mathbf{H}$ is a vertical matrix of order $q^2 \times q(q-1)$. After a column permutation, $\mathbf{H}$ has the form

$$\mathbf{H} = \left[\begin{array}{cccc|ccc|c|c} -\mathbf{U}_2 & -\mathbf{U}_3 & \cdots & -\mathbf{U}_q & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{U}_1 & \mathbf{O} & \cdots & \mathbf{O} & -\mathbf{U}_3 & \cdots & -\mathbf{U}_q & \mathbf{O} & \vdots \\ \mathbf{O} & \mathbf{U}_1 & \ddots & \vdots & \mathbf{U}_2 & \mathbf{O} & \mathbf{O} & & \mathbf{O} \\ \vdots & \ddots & \ddots & \mathbf{O} & \mathbf{O} & \ddots & \mathbf{O} & \cdots & -\mathbf{U}_q \\ \mathbf{O} & \cdots & \mathbf{O} & \mathbf{U}_1 & \mathbf{O} & \mathbf{O} & \mathbf{U}_2 & & \mathbf{U}_{q-1} \end{array}\right], \tag{6}$$

where $\mathbf{U}_k$ is a $q \times 2$ matrix, $k = 1, \ldots, q$. It can be verified that $\mathbf{H}$ in (6) has full column rank for $[\mathbf{U}_1|\cdots|\mathbf{U}_q] = [\mathbf{I}_q|\mathbf{I}_q]$ when q is odd and for $[\mathbf{U}_1|\cdots|\mathbf{U}_q] = \left[\begin{array}{cc|cc} \mathbf{I}_{q-1} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{q-1} \\ \mathbf{0}' & 1 & 1 & \mathbf{0}' \end{array}\right]$ when q is even. Hence, for each q we can specify a matrix $[\mathbf{U}_1|\cdots|\mathbf{U}_q]$ such that $\mathbf{H}$ has full column rank. Using [3, Theorem 5.A.2] as above (with the determinant of a $q(q-1) \times q(q-1)$ submatrix of $\mathbf{H}$ as analytic function of the elements of $[\mathbf{U}_1|\cdots|\mathbf{U}_q]$), it follows that $\mathbf{H}$ has full column rank with probability 1 if $q > p = 2$.

For $q > p \geqslant 3$, the matrix $\mathbf{H}$ is horizontal. Unfortunately, we were unable to give a particular $[\mathbf{U}_1|\cdots|\mathbf{U}_q]$ for which $\mathbf{H}$ has full row rank. But it can be verified numerically that this is true for any randomly sampled matrix $[\mathbf{U}_1|\cdots|\mathbf{U}_q]$. Hence, it follows from Fisher [3, Theorem 5.A.2] that $\mathbf{H}$ has full row rank with probability 1 if $q > p \geqslant 3$. This completes the proof.

Appendix B. Random $4 \times 4 \times 3$ arrays have rank 6 with positive probability

Let the elements of a real-valued $4 \times 4 \times 3$ array be randomly sampled from a 48-dimensional continuous distribution. We denote the three 4×4 slices of the array by **X**, **Y** and **Z**. We assume that the array consisting of slices **X** and **Y** has rank 4. This occurs with positive probability, because the rank of random $p \times p \times 2$ arrays equals p with positive probability (see [15]). We will show that in this case, a rank-6 decomposition of the $4 \times 4 \times 3$ array exists with positive probability.

Let the rank-4 decomposition of **X** and **Y** be given by $\mathbf{X} = \mathbf{A}_4\mathbf{C}_4^{(1)}\mathbf{B}_4'$ and $\mathbf{Y} = \mathbf{A}_4\mathbf{C}_4^{(2)}\mathbf{B}_4'$, where $\mathbf{A}_4$ and $\mathbf{B}_4$ are 4×4 nonsingular, and $\mathbf{C}_4^{(1)}$ and $\mathbf{C}_4^{(2)}$ are 4×4 diagonal and nonsingular. By premultiplying the three slices by $\mathbf{A}_4^{-1}$ and postmultiplying by $(\mathbf{B}_4')^{-1}$, the rank of the array remains the same. This implies that, for the rank-6 decomposition, we may assume that $\mathbf{A} = [\mathbf{I}_4 \quad \mathbf{x} \quad \mathbf{y}]$, $\mathbf{B} = [\mathbf{I}_4 \quad \mathbf{u} \quad \mathbf{v}]$ and

$$\mathbf{C} = \begin{bmatrix} * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & * \end{bmatrix}, \tag{7}$$

where an $*$ indicates a nonzero element. The first row of **C** contains the diagonal elements of $\mathbf{C}_4^{(1)}$ and the second row the diagonal elements of $\mathbf{C}_4^{(2)}$.

We need to show that also the third slice **Z** can be fitted with the rank-6 decomposition above (with positive probability). Since the first four columns of the component matrices can only be used to adjust the diagonal elements, the question boils down to: is the probability that a random 4×4 matrix **Z** can be made of rank 2 by adjusting only its diagonal elements, positive? Below, we show that this is indeed the case.

Denote the columns of **Z** by $\mathbf{z}_i$, $i = 1, 2, 3, 4$. We need the following lemma.

Lemma 1. *If there holds that*

$$\mathbf{z}_1 = \alpha\mathbf{z}_2 + \beta\mathbf{z}_3 \quad \textit{and} \quad \mathbf{z}_1 = \gamma\mathbf{z}_3 + \delta\mathbf{z}_4, \tag{8}$$

where $\alpha, \beta, \gamma, \delta$ *are nonzero, then* $\text{rank}(\mathbf{Z}) \leqslant 2$.

Proof. From the second equation in (8) it follows that $\mathbf{z}_1$ lies in the column space of $\mathbf{z}_3$ and $\mathbf{z}_4$. Combining the two equations in (8), we obtain

$$\mathbf{z}_2 = \frac{1}{\alpha}\mathbf{z}_1 - \frac{\beta}{\alpha}\mathbf{z}_3 = \frac{\gamma - \beta}{\alpha}\mathbf{z}_3 + \frac{\delta}{\alpha}\mathbf{z}_4, \tag{9}$$

which shows that also $\mathbf{z}_2$ lies in the column space of $\mathbf{z}_3$ and $\mathbf{z}_4$. This implies that **Z** has at most rank 2. □

To show that **Z** can be reduced to rank 2 with positive probability, we write the equations in (8) as follows:

$$\mathbf{F}_1 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} z_{11} \\ z_{41} \end{pmatrix}, \quad \text{where } \mathbf{F}_1 = \begin{bmatrix} z_{12} & z_{13} \\ z_{42} & z_{43} \end{bmatrix}, \tag{10}$$

$$z_{21} = \alpha z_{22} + \beta z_{23}, \tag{11}$$

$$z_{31} = \alpha z_{32} + \beta z_{33}, \tag{12}$$

$$\mathbf{F}_2 \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} z_{21} \\ z_{31} \end{pmatrix}, \quad \text{where } \mathbf{F}_2 = \begin{bmatrix} z_{23} & z_{24} \\ z_{33} & z_{34} \end{bmatrix}, \tag{13}$$

$$z_{11} = \gamma z_{13} + \delta z_{14}, \tag{14}$$

$$z_{41} = \gamma z_{43} + \delta z_{44}. \tag{15}$$

First, we determine the values of z_{11} and z_{33}. The matrix $\mathbf{F}_1$ is nonsingular with probability 1. We substitute the solutions for α and β from (10) into (12), and rewrite the result to obtain

$$z_{33} = \frac{m_1 - m_2 z_{11}}{m_3 - m_4 z_{11}}, \tag{16}$$

where $m_1 = z_{31}(z_{12}z_{43} - z_{42}z_{13}) + z_{13}z_{32}z_{41}$, $m_2 = z_{32}z_{43}$, $m_3 = z_{12}z_{41}$ and $m_4 = z_{42}$. We also assume that $\mathbf{F}_2$ is nonsingular. We substitute the solutions for γ and δ from (13) into (14), and rewrite the result to obtain

$$z_{33} = \frac{n_1 - n_2 z_{11}}{n_3 - n_4 z_{11}}, \tag{17}$$

where $n_1 = z_{13}(z_{21}z_{34} - z_{24}z_{31}) + z_{31}z_{23}z_{14}$, $n_2 = z_{23}z_{34}$, $n_3 = z_{21}z_{14}$ and $n_4 = z_{24}$. By equating (16) and (17), we obtain the following equation for z_{11}:

$$(m_2n_4 - n_2m_4)z_{11}^2 + (n_2m_3 - n_3m_2 + n_1m_4 - m_1n_4)z_{11} + (m_1n_3 - n_1m_3) = 0. \tag{18}$$

If the second degree polynomial in (18) has real roots, then we take z_{11} as one of them. The value of z_{33} can be determined from (16) or (17). After (10) and (13) have been solved, the values of z_{22} and z_{44} follow from (11) and (15). With these diagonal elements, the matrix $\mathbf{Z}$ will have rank 2. When this approach is applied to

$$\mathbf{Z} = \begin{bmatrix} 0 & 2 & 1 & -1 \\ -1 & 2 & 1 & -3 \\ 2 & -1 & 1 & 8 \\ 3 & 4 & 1 & 3 \end{bmatrix}, \tag{19}$$

the solution is: $z_{11} = 1$, $z_{22} = 0$, $z_{33} = -3$ and $z_{44} = 1$. There holds $\alpha = 1$, $\beta = -1$, $\gamma = 2$ and $\delta = 1$. It can be verified that our approach also works in a small surrounding of $\mathbf{Z}$ in (19). This completes the proof.

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