Real-valued 4×3×3 arrays have rank 5 with positive probability

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Introduction

We consider a real-valued 4×3×3 array of which the elements are drawn from a 36-dimensional continuous distribution *P*. We assume that *P*(*A*) = 0 if and only if *L*(*A*) = 0, where *L* denotes the Lebesgue measure and *A* is an arbitrary Borel set in $\Re^{36}$. The three 4×3 slices of the array are denoted by **X**, **Y** and **Z**. We know that, with probability 1, the three-way rank of the array is either 5 or 6; see Ten Berge and Stegeman (2004, Table 2). Here, we will show that rank 5 occurs with positive probability. It is not yet known whether rank 6 occurs with positive probability or not.

Our result is obtained by showing that all arrays randomly sampled from a small 36-dimensional environment of a particular 4×3×3 array, have a full rank-5 decomposition. Let **X**, **Y** and **Z** be randomly sampled as explained above. Ten Berge and Kiers (1999) have shown that, with probability 1, there exist nonsingular matrices **S** and **T** such that

$$\mathbf{SXT} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{SYT} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{SZT} = \begin{bmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \\ f_4 & g_4 & h_4 \end{bmatrix}, \tag{1}$$

where the last slice can be treated as randomly sampled from a 12-dimensional continuous distribution. Hence, without loss of generality we may assume that the array is of the form as in (1), i.e.

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \\ f_4 & g_4 & h_4 \end{bmatrix}. \tag{2}$$

Moreover, since all arrays randomly sampled from a small 36-dimensional environment of (**X**,**Y**,**Z**) can be transformed to the form (1), we only need to show that a full rank-5 decomposition exists for **X** and **Y** as in (2) and a small 12-dimensional environment of a particular **Z**. For this, we will take

$$\mathbf{Z} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 4 & 7 \end{bmatrix}. \tag{3}$$

Construction of a rank-5 decomposition

We start with the array in (2) and show how a rank-5 decomposition can be obtained. This is done by adding a fifth row to the slices in (2) and using the approach of Ten Berge (2004) to find a rank-5 decomposition of a 5×3×3 array. We denote our 5×3×3 array by

$$\tilde{\mathbf{X}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ x_1 & x_2 & x_3 \end{bmatrix}, \quad \tilde{\mathbf{Y}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ y_1 & y_2 & y_3 \end{bmatrix}, \quad \tilde{\mathbf{Z}} = \begin{bmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \\ f_4 & g_4 & h_4 \\ z_1 & z_2 & z_3 \end{bmatrix}, \tag{4}$$

where we may choose the values of $\mathbf{x} = (x_1, x_2, x_3)'$, $\mathbf{y} = (y_1, y_2, y_3)'$ and $\mathbf{z} = (z_1, z_2, z_3)'$. Let $\mathbf{f} = (f_1, f_2, f_3, f_4)'$, $\mathbf{g} = (g_1, g_2, g_3, g_4)'$ and $\mathbf{h} = (h_1, h_2, h_3, h_4)'$. It can be seen that we may set $x_1 = x_2 = x_3 = y_3 = 0$ without loss of generality. Indeed, subtracting from the fifth row in each slice x_1 times the first row, x_2 times the second row, x_3 times the third row and y_3 times the fourth row yields an array with the same rank as (4). Somewhat abusing the notation, the remaining elements on the fifth row will be denoted as $\mathbf{y} = (y_1, y_2)'$ and $\mathbf{z} = (z_1, z_2, z_3)'$. Hence, we consider the following array:

$$\tilde{\mathbf{X}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{Y}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ y_1 & y_2 & 0 \end{bmatrix}, \quad \tilde{\mathbf{Z}} = \begin{bmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \\ f_4 & g_4 & h_4 \\ z_1 & z_2 & z_3 \end{bmatrix}. \tag{5}$$

The following rank-5 decomposition of (5) will be derived:

$$\tilde{\mathbf{X}} = \mathbf{A}\,\mathbf{I}_5\,\mathbf{B}', \qquad \tilde{\mathbf{Y}} = \mathbf{A}\,\mathbf{C}\,\mathbf{B}', \qquad \tilde{\mathbf{Z}} = \mathbf{A}\,\mathbf{D}\,\mathbf{B}', \tag{6}$$

with **A** (5×5), **B** (3×5) and **C** and **D** diagonal matrices. If the fifth row of **A** is deleted, a rank-5 decomposition of the 4×3×3 array in (2) is the result.

Next, we show how to derive the component matrices **A**, **B**, **C** and **D**. We assume that **A** is nonsingular. It can be seen that $\mathbf{C}\mathbf{A}^{-1}\tilde{\mathbf{X}} - \mathbf{A}^{-1}\tilde{\mathbf{Y}} = \mathbf{A}^{-1}\tilde{\mathbf{Z}} - \mathbf{D}\mathbf{A}^{-1}\tilde{\mathbf{X}} = \mathbf{O}$. Hence, the *j*-th row $\mathbf{a}_j'$ of $\mathbf{A}^{-1}$ should satisfy $\mathbf{a}_j'(c_j\tilde{\mathbf{X}} - \tilde{\mathbf{Y}}) = \mathbf{a}_j'(\tilde{\mathbf{Z}} - d_j\tilde{\mathbf{X}}) = \mathbf{0}'$, where c_j and d_j are the diagonal elements of **C** and **D**, respectively. Set the first element of $\mathbf{a}_j$ to 1. From (5) it follows that $\mathbf{a}_j'(c_j\tilde{\mathbf{X}} - \tilde{\mathbf{Y}}) = \mathbf{0}'$ is equivalent to

$$\mathbf{a}_j = \begin{pmatrix} 1 \\ c_j - \beta_j y_1 \\ c_j^2 - \beta_j y_1 c_j - \beta_j y_2 \\ c_j^3 - \beta_j y_1 c_j^2 - \beta_j y_2 c_j \\ \beta_j \end{pmatrix}, \tag{7}$$

for some scalar β_j. It remains to satisfy $\mathbf{a}_j'(\tilde{\mathbf{Z}} - d_j\tilde{\mathbf{X}}) = \mathbf{0}'$, i.e. $\mathbf{a}_j$ has to be orthogonal to the three columns of $\tilde{\mathbf{Z}} - d_j\tilde{\mathbf{X}}$. For $\mathbf{a}_j$ in (7), this is equivalent to the vector $\begin{pmatrix} 1 \\ \beta_j \end{pmatrix}$ being orthogonal to the columns of the 2×3 matrix

$$\mathbf{W}_j = \begin{bmatrix} F^{(1)}(c_j,d_j) & G^{(1)}(c_j,d_j) & H^{(1)}(c_j,d_j) \\ F^{(2)}(c_j) & G^{(2)}(c_j,d_j) & H^{(2)}(c_j,d_j) \end{bmatrix}, \tag{8}$$

where the expressions for the elements of $\mathbf{W}_j$ are given in the Appendix. For the vector $\begin{pmatrix} 1 \\ \beta_j \end{pmatrix}$ to be orthogonal to the columns of $\mathbf{W}_j$, we must have rank($\mathbf{W}_j$) = 1. We ensure this by choosing c_j and d_j such that

$$\det \begin{bmatrix} F^{(1)}(c_j,d_j) & G^{(1)}(c_j,d_j) \\ F^{(2)}(c_j) & G^{(2)}(c_j,d_j) \end{bmatrix} = \det \begin{bmatrix} F^{(1)}(c_j,d_j) & H^{(1)}(c_j,d_j) \\ F^{(2)}(c_j) & H^{(2)}(c_j,d_j) \end{bmatrix} = 0, \tag{9}$$

but not $F^{(1)}(c_j,d_j) = F^{(2)}(c_j) = 0$. The first determinant in (9) equals

$$e_3 c_j^3 + (e_{21} + e_{22} d_j) c_j^2 + (e_{11} + e_{12} d_j) c_j + (e_{01} + e_{02} d_j + e_{03} d_j^2), \tag{10}$$

where the coefficients e_k depend on **f**, **g**, **h**, **y** and **z**. The expressions for e_k are given in the Appendix. The second determinant in (9) equals

$$\tilde{e}_3\, c_j^3 + (\tilde{e}_{21} + \tilde{e}_{22}\, d_j)\, c_j^2 + (\tilde{e}_{11} + \tilde{e}_{12}\, d_j + \tilde{e}_{13}\, d_j^2)\, c_j + (\tilde{e}_{01} + \tilde{e}_{02}\, d_j + \tilde{e}_{03}\, d_j^2)\,, \quad (11)$$

where the coefficients $\tilde{e}_k$ depend on **f**, **g**, **h**, **y** and **z**. The expressions for $\tilde{e}_k$ are given in the Appendix. The following lemma specifies the solutions (c_j, d_j) for which $F^{(1)}(c_j, d_j) = F^{(2)}(c_j) = 0$.

<u>Lemma 1</u>: The expression $F^{(1)}(c_j, d_j) = F^{(2)}(c_j) = 0$ is equivalent to:

$$-(f_4 y_1)\, c_j^2 - (f_3 y_1 + f_4 y_2)\, c_j + (z_1 - f_3 y_2 - f_2 y_1) = 0\,, \quad (12)$$

$$d_j = f_4\, c_j^3 + f_3\, c_j^2 + f_2\, c_j + f_1\ . \quad (13)$$

Proof. It can be seen (see Appendix) that (12) is equivalent to $F^{(2)}(c_j) = 0$ and (13) is equivalent to $F^{(1)}(c_j, d_j) = 0$. This completes the proof. □

Next, we determine (c_j, d_j) which satisfy both (10) and (11). We set $y_1 = 0$. Then $e_{03} = \tilde{e}_{13} = 0$ (see Appendix) and d_j can be determined from (10) as

$$d_j = -\left(\frac{e_3\, c_j^3 + e_{21}\, c_j^2 + e_{11}\, c_j + e_{01}}{e_{22}\, c_j^2 + e_{12}\, c_j + e_{02}} \right). \quad (12)$$

Substituting (12) into (11) yields that c_j can be found as a root of a 7-th degree polynomial. We denote this polynomial by

$$Q(c) = q_7\, c^7 + q_6\, c^6 + q_5\, c^5 + q_4\, c^4 + q_3\, c^3 + q_2\, c^2 + q_1\, c + q_0\,. \quad (13)$$

The coefficients q_k depend only on **f**, **g**, **h**, **y** and **z**. Their expressions are given in the Appendix. Since we need five solutions for c_j, the polynomial Q must have at least five real roots. Because $y_1 = 0$, there is only one solution (c_j, d_j) for which $F^{(1)}(c_j, d_j) = F^{(2)}(c_j) = 0$ (see Lemma 1). The c_j of this flawed solution is also a root of Q. Therefore, Q must have seven real roots. One root is discarded and five out of the six remaining roots are used as c_j. This is again the partial uniqueness result obtained by Ten Berge (2004). Once the c_j and d_j are known, the scalars β_j can be chosen as

$-F^{(1)}(c_j, d_j)/F^{(2)}(c_j)$. Then $\mathbf{A}^{-1}$ follows from (7) and **B** can be determined from $\mathbf{B}' = \mathbf{A}^{-1}\tilde{\mathbf{X}}$.

Hence, for given vectors **f**, **g**, **h**, the problem is to choose **y** and **z** (under the restriction $y_1 = 0$) such that the polynomial Q in (13) has seven real roots.

Rank 5 occurs with positive probability

We have applied the procedure above to the 4×3×3 array in (2) with **Z** as in (3). For

$$\mathbf{y} = \begin{pmatrix} 0 \\ -0.42 \end{pmatrix}, \qquad \mathbf{z} = \begin{pmatrix} -0.06 \\ -0.15 \\ -0.27 \end{pmatrix}, \tag{14}$$

the polynomial Q has the following seven real roots: -9.85, -1.46, -0.86, -0.69, -0.68, 0.31 and 3.46. From Lemma 1 it follows that the flawed solution for c_j is given by

$$c_j = \frac{z_1 - f_3 y_2}{f_4 y_2} = -0.86. \tag{15}$$

Hence, six real roots of Q remain and a rank-5 decomposition can be constructed as above.

Next, we show that a full rank-5 decomposition is also possible in a small environment of **Z**, where **y** and **z** remain the same. Define

$$S_8 = \{(q_0, q_1, \ldots, q_7) : Q \text{ has seven real roots}\}. \tag{16}$$

Since Q has a unique set of seven roots (for $q_7 \neq 0$), we may write

$$Q(c) = \alpha(\lambda_1 - c)(\lambda_2 - c)(\lambda_3 - c)(\lambda_4 - c)(\lambda_5 - c)(\lambda_6 - c)(\lambda_7 - c), \tag{17}$$

where α is a scaling parameter and λ_i are the roots of Q. By equating (13) and (17), it can be verified that there exists a continuous mapping from $(\alpha, \lambda_1, \lambda_2, \ldots, \lambda_7)$ to $(q_0, q_1, \ldots, q_7)$. Moreover, since Q has a unique set of seven roots (for $q_7 \neq 0$), this mapping is one-to-one up to a permutation of $(\lambda_1, \lambda_2, \ldots, \lambda_7)$. This implies that the set S_8 has positive 8-dimensional volume. The boundary points of S_8 are those $(q_0, q_1, \ldots, q_7)$ for which Q has at least two identical real roots. Then an aribitrary close approximation by a polynomial with one pair of complex roots is possible, where the imaginary parts of the complex roots are close to zero. Hence, S_8 is a closed set. If Q has seven distinct real roots for $(q_0, q_1, \ldots, q_7)$, then $(q_0, q_1, \ldots, q_7)$ is an interior point of the set S_8, and within a small environment of $(q_0, q_1, \ldots, q_7)$ the polynomial Q also has seven real roots. Since the coefficients $(q_0, q_1, \ldots, q_7)$ are continuous functions of **f**, **g** and **h**, it follows that in a small environment of **Z** in (3), with **y** and **z** as in (14), the polynomial Q will still have seven real roots. Therefore, a full rank-5 decomposition is possible in a small environment of **Z**. This shows that for real-valued 4×3×3 arrays of which the elements are randomly sampled from a continuous distribution, rank 5 occurs with positive probability.

References

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Appendix

The expressions for the elements of the matrix $\mathbf{W}_j$ in (8) are as follows:

$$F^{(1)}(c,d) = f_4 c^3 + f_3 c^2 + f_2 c + (f_1 - d),$$

$$G^{(1)}(c,d) = g_4 c^3 + g_3 c^2 + (g_2 - d)c + g_1,$$

$$H^{(1)}(c,d) = h_4 c^3 + (h_3 - d)c^2 + h_2 c + h_1,$$

$$F^{(2)}(c) = -(f_4 y_1)c^2 - (f_3 y_1 + f_4 y_2)c + (z_1 - f_3 y_2 - f_2 y_1),$$

$$G^{(2)}(c,d) = -(g_4 y_1)c^2 - (g_3 y_1 + g_4 y_2)c + [z_2 - g_3 y_2 - (g_2 - d)y_1],$$

$$H^{(2)}(c,d) = -(h_4 y_1)c^2 - [(h_3 - d)y_1 + h_4 y_2]c + [z_3 - (h_3 - d)y_2 - h_2 y_1].$$

The expressions for the coefficients e_k in (10) are as follows:

$e_3 = (f_4 z_2 - g_4 z_1)$,

$e_{21} = (f_3 z_2 - g_3 z_1) - (f_1 g_4 - g_1 f_4) y_1 - (f_2 g_4 - g_2 f_4) y_2$,

$e_{22} = g_4 y_1 - f_4 y_2$,

$e_{11} = (f_2 z_2 - g_2 z_1) + (f_3 g_2 - g_3 f_2) y_2 - (f_1 g_3 - g_1 f_3) y_1 - (f_1 g_4 - g_1 f_4) y_2$,

$e_{01} = (f_1 z_2 - g_1 z_1) + (f_2 g_1 - g_2 f_1) y_1 + (f_3 g_1 - g_3 f_1) y_2$,

$e_{02} = (f_1 + g_2) y_1 + g_3 y_2 - z_2$,

$e_{03} = -y_1$.

The expressions for the coefficients $\tilde{e}_k$ in (11) are as follows:

$\tilde{e}_3 = (f_4 z_3 - h_4 z_1)$,

$\tilde{e}_{21} = (f_3 z_3 - h_3 z_1) - (f_1 h_4 - h_1 f_4) y_1 - (f_2 h_4 - h_2 f_4) y_2$,

$\tilde{e}_{22} = z_1 + h_4 y_1$,

$\tilde{e}_{11} = (f_2 z_3 - h_2 z_1) + (f_3 h_2 - h_3 f_2) y_2 - (f_1 h_3 - h_1 f_3) y_1 - (f_1 h_4 - h_1 f_4) y_2$,

$\tilde{e}_{12} = (f_1 + h_3) y_1 + (f_2 + h_4) y_2$,

$\tilde{e}_{13} = -y_1$,

$\tilde{e}_{01} = (f_1 z_3 - h_1 z_1) + (f_2 h_1 - h_2 f_1) y_1 + (f_3 h_1 - h_3 f_1) y_2$,

$\tilde{e}_{02} = h_2 y_1 + (f_1 + h_3) y_2 - z_3$,

$\tilde{e}_{03} = -y_2$.

The expressions for the coefficients q_k in (13) are as follows:

$$q_7 = \tilde{e}_3 e_{22}^2 - \tilde{e}_{22} e_3 e_{22},$$

$$q_6 = 2\tilde{e}_3 e_{22} e_{12} + \tilde{e}_{21} e_{22}^2 - \tilde{e}_{22}(e_3 e_{12} + e_{21} e_{22}) - \tilde{e}_{12} e_3 e_{22} + \tilde{e}_{03} e_3^2,$$

$$q_5 = \tilde{e}_3(2e_{22} e_{02} + e_{12}^2) + 2\tilde{e}_{21} e_{22} e_{12} - \tilde{e}_{22}(e_3 e_{02} + e_{21} e_{12} + e_{11} e_{22}) + \tilde{e}_{11} e_{22}^2$$
$$- \tilde{e}_{12}(e_3 e_{12} + e_{21} e_{22}) - \tilde{e}_{02} e_3 e_{22} + 2\tilde{e}_{03} e_{21} e_3,$$

$$q_4 = 2\tilde{e}_3 e_{12} e_{02} + \tilde{e}_{21}(2e_{22} e_{02} + e_{12}^2) - \tilde{e}_{22}(e_{21} e_{02} + e_{11} e_{12} + e_{01} e_{22}) + 2\tilde{e}_{11} e_{22} e_{12}$$
$$- \tilde{e}_{12}(e_3 e_{02} + e_{21} e_{12} + e_{11} e_{22}) + \tilde{e}_{01} e_{22}^2 - \tilde{e}_{02}(e_3 e_{12} + e_{21} e_{22}) + \tilde{e}_{03}(e_{21}^2 + 2e_{11} e_3),$$

$$q_3 = \tilde{e}_3 e_{02}^2 + 2\tilde{e}_{21} e_{12} e_{02} - \tilde{e}_{22}(e_{11} e_{02} + e_{01} e_{12}) + \tilde{e}_{11}(2e_{22} e_{02} + e_{12}^2) - \tilde{e}_{12}(e_{21} e_{02} + e_{11} e_{12} + e_{01} e_{22})$$
$$+ 2\tilde{e}_{01} e_{22} e_{12} - \tilde{e}_{02}(e_3 e_{02} + e_{21} e_{12} + e_{11} e_{22}) + 2\tilde{e}_{03}(e_{01} e_3 + 2e_{11} e_{21}),$$

$$q_2 = \tilde{e}_{21} e_{02}^2 - \tilde{e}_{22} e_{01} e_{02} + 2\tilde{e}_{11} e_{12} e_{02} - \tilde{e}_{12}(e_{11} e_{02} + e_{01} e_{12}) + \tilde{e}_{01}(2e_{22} e_{02} + e_{12}^2)$$
$$- \tilde{e}_{02}(e_{21} e_{02} + e_{11} e_{12} + e_{01} e_{22}) + \tilde{e}_{03}(2e_{01} e_{21} + e_{11}^2),$$

$$q_1 = \tilde{e}_{11} e_{02}^2 - \tilde{e}_{12} e_{01} e_{02} + 2\tilde{e}_{01} e_{12} e_{02} - \tilde{e}_{02}(e_{11} e_{02} + e_{01} e_{12}) + 2\tilde{e}_{03} e_{01} e_{11},$$

$$q_0 = \tilde{e}_{01} e_{02}^2 - \tilde{e}_{02} e_{01} e_{02} + \tilde{e}_{03} e_{01}^2.$$