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ON UNIQUENESS OF THE nTH ORDER TENSOR DECOMPOSITION INTO RANK-1 TERMS WITH LINEAR INDEPENDENCE IN ONE MODE*

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Abstract. We study uniqueness of the decomposition of an nth order tensor (also called n-way array) into a sum of R rank-1 terms (where each term is the outer product of n vectors). This decomposition is also known as Parafac or Candecomp, and a general uniqueness condition for $n = 3$ has been obtained by Kruskal in 1977 [*Linear Algebra Appl.*, 18 (1977), pp. 95–138]. More recently, Kruskal's uniqueness condition has been generalized to $n \geq 3$, and less restrictive uniqueness conditions have been obtained for the case where the vectors of the rank-1 terms are linearly independent in (at least) one of the n modes. For this case, only $n = 3$ and $n = 4$ have been studied. We generalize these results by providing a framework of analysis for arbitrary $n \geq 3$. Our results include necessary, sufficient, necessary and sufficient, and generic uniqueness conditions. For the sufficient uniqueness conditions, the rank of a matrix needs to be checked. The generic uniqueness conditions have the form of a bound on R in terms of the dimensions of the tensor to be decomposed.

Key words. tensor decomposition, uniqueness, Parafac, Candecomp

AMS subject classifications. 15A18, 15A69

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1. Introduction.

Tensors of order n are defined on the outer product of n linear spaces, $\mathcal{S}_\ell$, $1 \leq \ell \leq n$. Once bases of spaces $\mathcal{S}_\ell$ are fixed, they can be represented by n-way arrays. For simplicity, tensors are usually assimilated with their array representation.

We consider the nth order tensor decomposition of the form

$$\underline{\mathbf{X}} = \sum_{r=1}^{R} \mathbf{a}_r^{(1)} \circ \mathbf{a}_r^{(2)} \circ \cdots \circ \mathbf{a}_r^{(n)} \,, \tag{1.1}$$

where $\underline{\mathbf{X}} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_n}$ is an nth order tensor (or n-way array), $\mathbf{a}_r^{(j)} \in \mathbb{R}^{I_j}$ are vectors, and $\circ$ denotes the outer vector product. For vectors $\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)}$, the outer vector product $\mathbf{a}^{(1)} \circ \cdots \circ \mathbf{a}^{(n)}$ is an nth order tensor with entries $a_{i_1}^{(1)} a_{i_2}^{(2)} \cdots a_{i_n}^{(n)}$. We refer to $\underline{\mathbf{X}}$ in (1.1) as having n *modes*, and the j in $\mathbf{a}_r^{(j)}$ corresponds to mode j. Note that when the modes of $\underline{\mathbf{X}}$ are permuted in (1.1), the vectors $\mathbf{a}_r^{(j)}$ are permuted identically.

We will denote vectors as $\mathbf{x}$, matrices (2nd order tensors, 2-way arrays) as $\mathbf{X}$, and higher-order tensors (multiway arrays) as $\underline{\mathbf{X}}$. We use $\otimes$ to denote the usual Kronecker product, and $\odot$ denotes the (columnwise) Khatri–Rao product, i.e., for matrices $\mathbf{X}$ and $\mathbf{Y}$ with R columns, $\mathbf{X} \odot \mathbf{Y} = [\mathbf{x}_1 \otimes \mathbf{y}_1|, \ldots, |\mathbf{x}_R \otimes \mathbf{y}_R]$. The transpose of $\mathbf{X}$ is denoted as $\mathbf{X}^T$, and $\text{diag}(\mathbf{x})$ denotes the diagonal matrix with the entries in vector

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$\mathbf{x}$ on its diagonal. We refer to a matrix as having *full column rank* if its rank equals its number of columns. Analogously, a matrix has *full row rank* if its rank equals its number of rows.

An nth order tensor has rank 1 if it can be written as the outer product of n vectors. The rank of an nth order tensor $\underline{\mathbf{X}}$ is defined as the smallest number of rank-1 tensors whose sum equals $\underline{\mathbf{X}}$. Hence, (1.1) decomposes $\underline{\mathbf{X}}$ into R rank-1 terms. Hitchcock [12, 13] introduced tensor rank and the related tensor decomposition (1.1). The same decomposition was proposed independently by Carroll and Chang [3] and Harshman [11] for component analysis of nth order tensors. They named it Candecomp and Parafac, respectively.

For a given nth order tensor and number R of rank-1 components, a best fitting decomposition (1.1) is usually found by an iterative algorithm. The most well-known algorithm is alternating least squares. A comparison of algorithms for $n = 3$ can be found in Tomasi and Bro [38]. Note that a best fitting decomposition is a best rank-R approximation of the tensor.

Real-valued applications of tensor decompositions occur in psychology and chemistry; see Kroonenberg [20], Kiers and Van Mechelen [17], and Smilde, Bro, and Geladi [29]. Complex-valued tensor decompositions are used in, e.g., signal processing and telecommunications research; see Sidiropoulos, Giannakis, and Bro [27], Sidiropoulos, Bro, and Giannakis [28], and De Lathauwer and Castaing [7]. Applications of the tensor decomposition (1.1) for $n \geq 4$ can be found in chemometrics (Durell et al. [10]), and neuroimaging (Andersen and Rayens [2] and Mørup et al. [24]). Also, for $n = 4$, the tensor decomposition (1.1) describes the basic structure of 4th order cumulants of multivariate data on which a lot of algebraic methods for independent component analysis (ICA) are based (Comon [4], De Lathauwer, De Moor, and Vandewalle [5], and Hyvärinen, Karhunen, and Oja [14]). ICA algorithms explicitly using (1.1) can be found in De Lathauwer, Castaing, and Cardoso [8] (for $n = 4$) and in Karfoul, Albera, and De Lathauwer [16] (for $n = 6$). For a general overview of applications of the decomposition (1.1) and related decompositions, see Kolda and Bader [18] or Acar and Yener [1].

A drawback of computing a best fitting tensor decomposition (1.1) is that an optimal solution may not exist. Indeed, a tensor may not have a best rank-R approximation. This is due to the fact that the set of tensors of rank at most R is not closed; see De Silva and Lim [9]. In such cases, diverging components (i.e., close to linear dependence and large in magnitude) occur while running an iterative algorithm designed to find a best rank-R approximation; see Krijnen, Dijkstra, and Stegeman [19]. This phenomenon is also known as "degeneracy"; see Kruskal, Harshman, and Lundy [22], and Stegeman [30, 31, 32, 33]. This problem can be fixed by including interaction terms in the decomposition; see Stegeman and De Lathauwer [37] for the case $n = 3$ and $I_3 = 2$, and Rocci and Giordani [25] for the case $n = 3$ and $R = 2$.

An attractive feature of the decomposition (1.1) is that the vectors $\mathbf{a}_r^{(j)}$ are unique under mild conditions. We define uniqueness of (1.1) as follows. Let $\mathbf{A}^{(j)} = [\mathbf{a}_1^{(j)}|\mathbf{a}_2^{(j)}|\ldots|\mathbf{a}_R^{(j)}]$ denote the jth component matrix. Hence, matrix $\mathbf{A}^{(j)}$ has size $I_j \times R$. We denote an nth order decomposition (1.1) as $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$.

Definition 1.1. *The decomposition* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ *is called* unique up to permutation and scaling *if any alternative decomposition* $(\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(n)})$ *satisfies* $\mathbf{B}^{(j)} = \mathbf{A}^{(j)}\mathbf{\Pi}\,\mathbf{\Lambda}_j$, $j = 1, \ldots, n$, *with* $\mathbf{\Pi}$ *an* $R \times R$ *permutation matrix, and* $\mathbf{\Lambda}_j$ *nonsingular diagonal matrices such that* $\prod_{j=1}^n \mathbf{\Lambda}_j = \mathbf{I}_R$. □

Hence, an nth order decomposition is unique up to permutation and scaling if the only ambiguities it contains are the permutation of the R rank-1 components, and the scaling of the n vectors constituting each rank-1 component.

The classical uniqueness condition for $n = 3$ is due to Kruskal [21]. Kruskal's condition relies on a particular concept of matrix rank that he introduced, which has been named k-rank after him. Specifically, the k-rank of a matrix is the largest number x such that every subset of x columns of the matrix is linearly independent. We denote the k-rank of a matrix $\mathbf{A}$ as $k_{\mathbf{A}}$. For a decomposition $(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{A}^{(3)})$, Kruskal [21] proved that

$$2\,R + 2 \leq k_{\mathbf{A}^{(1)}} + k_{\mathbf{A}^{(2)}} + k_{\mathbf{A}^{(3)}} \tag{1.2}$$

is a sufficient condition for uniqueness up to permutation and scaling. A more condensed and accessible proof of (1.2) was given by Stegeman and Sidiropoulos [36]. Kruskal's uniqueness condition was generalized to $n \geq 3$ by Sidiropoulos and Bro [26]. For a decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ the uniqueness condition becomes

$$2\,R + (n-1) \leq \sum_{j=1}^{n} k_{\mathbf{A}^{(j)}} \,. \tag{1.3}$$

By comparing (1.2) and (1.3), it can be seen that the uniqueness condition becomes less restrictive as the order n increases. Indeed, when increasing n by one the right-hand side of (1.3) increases with an additional k-rank while the left-hand side increases by one only.

For $n = 3$ and $n = 4$, less restrictive uniqueness conditions have been obtained for the case where (at least) one of the component matrices $\mathbf{A}^{(j)}$ has rank R, i.e., the vectors $\mathbf{a}_r^{(j)}$, $r = 1, \ldots, R$, are linearly independent in (at least) one mode j. In this paper, we consider this case for arbitrary order $n \geq 3$, and prove generalizations of existing uniqueness conditions. The next section contains a roadmap of uniqueness results in this paper, and indicates the links with existing uniqueness results. The organization of this paper will be explained at the end of the next section.

2. Roadmap of uniqueness results. Here, we present an overview of both existing and our new uniqueness conditions for a decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ with $\mathrm{rank}(\mathbf{A}^{(n)}) = R$. Also, our generalization of the approaches of Jiang and Sidiropoulos [15] and De Lathauwer [6] is discussed. First, however, we introduce some definitions. A mode-j vector of an $I_1 \times I_2 \times \cdots \times I_n$ tensor is defined as an $I_j \times 1$ vector that is obtained by varying the jth index and keeping the other indices fixed. A mode-j matrix unfolding of a tensor is defined as a matrix containing all mode-j vectors as columns. For the decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ in (1.1), we define the mode-j matrix unfolding as

$$\left(\bigodot_{i \neq j}^{n} \mathbf{A}^{(i)} \right) (\mathbf{A}^{(j)})^T \,, \tag{2.1}$$

where $\bigodot$ denote a series of (columnwise) Khatri–Rao products.

We denote an alternative decomposition as $(\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(n)})$, and focus on equating the mode-n matrix unfoldings of the two decompositions:

$$(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})\,(\mathbf{A}^{(n)})^T = (\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})\,(\mathbf{B}^{(n)})^T \,. \tag{2.2}$$

For $n = 3$, a necessary uniqueness condition is that the Khatri–Rao product of any two component matrices must have full column rank; see Liu and Sidiropoulos [23]. In Lemma 3.1, we prove a generalization of this necessary uniqueness condition for arbitrary order $n \geq 3$. In particular, $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$ must have full column rank R. We assume this to be true.

Since $\mathbf{A}^{(n)}$ has rank R and we assume that $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$ has rank R, it follows that also the right-hand side of (2.2) has rank R. Hence, $\mathbf{B}^{(n)}$ has rank R and $(\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})$ has rank R. Moreover, $\mathbf{A}^{(n)}$ and $\mathbf{B}^{(n)}$ have the same column space, which is also true for $(\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})$ and $(\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})$.

Next, we prove that if there holds $\mathbf{B}^{(n)} = \mathbf{A}^{(n)}\mathbf{\Pi}\,\mathbf{\Lambda}_n$ for a permutation matrix $\mathbf{\Pi}$ and a nonsingular diagonal matrix $\mathbf{\Lambda}_n$, then $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ is unique up to permutation and scaling. Hence, if $\mathbf{A}^{(n)}$ is unique up to permutation and scaling, then this is true for the complete nth order decomposition. Our proof of this is along the following lines. Under uniqueness of $\mathbf{A}^{(n)}$, (2.2) becomes

$$(2.3)\quad (\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})\,(\mathbf{A}^{(n)})^T = (\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})\,\mathbf{\Lambda}_n\,\mathbf{\Pi}^T(\mathbf{A}^{(n)})^T\,.$$

Since $\mathbf{A}^{(n)}$ has full column rank, this implies

$$(2.4)\qquad (\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)}) = (\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})\,\mathbf{\Pi}\,\mathbf{\Lambda}_n^{-1}\,.$$

In Lemma 4.1, we show that this implies $\mathbf{B}^{(j)} = \mathbf{A}^{(j)}\mathbf{\Pi}\,\mathbf{\Lambda}_j$, $j = 1, \ldots, n-1$, for nonsingular diagonal matrices $\mathbf{\Lambda}_j$ such that $\prod_{j=1}^{n-1}\mathbf{\Lambda}_j = \mathbf{\Lambda}_n^{-1}$. By Definition 1.1, this implies uniqueness of the decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$. For $n = 3$, this is shown by Jiang and Sidiropoulos [15].

Hence, the key step is to show uniqueness of $\mathbf{A}^{(n)}$. For this, we make use of Kruskal's [21] permutation lemma, which is formulated as Lemma 2.1 below. Let $\omega(\cdot)$ denote the number of nonzero elements of a vector.

LEMMA 2.1 (permutation lemma). *Let $\mathbf{A}$ and $\mathbf{B}$ be two $I \times R$ matrices and let $k_{\mathbf{A}} \geq 1$. Suppose the following condition holds: for any vector $\mathbf{x}$ such that $\omega(\mathbf{x}^T\mathbf{B}) \leq R - \mathrm{rank}(\mathbf{B}) + 1$, we have $\omega(\mathbf{x}^T\mathbf{A}) \leq \omega(\mathbf{x}^T\mathbf{B})$. Then there exists a permutation matrix $\mathbf{\Pi}$ and a nonsingular diagonal matrix $\mathbf{\Lambda}$ such that $\mathbf{B} = \mathbf{A}\,\mathbf{\Pi}\,\mathbf{\Lambda}$.* ∎

As observed above, we have $\mathrm{rank}(\mathbf{B}^{(n)}) = R$ in any alternative decomposition. Hence, in order to conclude uniqueness of $\mathbf{A}^{(n)}$ by the permutation lemma it suffices to show that for any vector $\mathbf{x}$ such that $\omega(\mathbf{x}^T\mathbf{B}^{(n)}) \leq 1$ we have $\omega(\mathbf{x}^T\mathbf{A}^{(n)}) \leq \omega(\mathbf{x}^T\mathbf{B}^{(n)})$. Since $\mathbf{A}^{(n)}$ and $\mathbf{B}^{(n)}$ have the same column space, $\omega(\mathbf{x}^T\mathbf{B}^{(n)}) = 0$ implies $\omega(\mathbf{x}^T\mathbf{A}^{(n)}) = 0$. Hence, the condition of the permutation lemma becomes the following: for any vector $\mathbf{x}$ such that $\omega(\mathbf{x}^T\mathbf{B}^{(n)}) = 1$ we have $\omega(\mathbf{x}^T\mathbf{A}^{(n)}) \leq 1$.

Let $\mathbf{x}$ be a vector with $\omega(\mathbf{x}^T\mathbf{B}^{(n)}) = 1$. By (2.2), we have

$$(2.5)\qquad (\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})\,(\mathbf{A}^{(n)})^T\mathbf{x} = (\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})\,(\mathbf{B}^{(n)})^T\mathbf{x}\,.$$

Since $\omega(\mathbf{x}^T\mathbf{B}^{(n)}) = 1$, the right-hand side of (2.5) is proportional to one column of $(\mathbf{B}^{(1)} \odot \cdots \odot \mathbf{B}^{(n-1)})$, and can be written as $(\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_{n-1})$ for some vectors $\mathbf{f}_j$, $j = 1, \ldots, n-1$. We write $\mathbf{d} = (\mathbf{A}^{(n)})^T\mathbf{x}$. Since $\mathbf{A}^{(n)}$ has full column rank, we may treat $\mathbf{d}$ as an arbitrary vector. It follows that a sufficient condition for uniqueness of the nth order decomposition is as follows: for any vector $\mathbf{d} = (d_1, d_2, \ldots, d_r)^T$

$$(2.6)\quad (\mathbf{A}^{(1)} \odot \cdots \odot \mathbf{A}^{(n-1)})\,\mathbf{d} = (\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_{n-1}) \qquad \text{implies} \qquad \omega(\mathbf{d}) \leq 1\,.$$

In Theorem 4.2, we show that (2.6) is also necessary for uniqueness. For $n = 3$, this is shown by Jiang and Sidiropoulos [15].

Condition (2.6) is not easy to check. For $n = 3$, Jiang and Sidiropoulos [15] show that (2.6) is equivalent to

$$\mathbf{U}^{(2)} \begin{pmatrix} d_1 d_2 \\ d_1 d_3 \\ \vdots \\ d_{R-1} d_R \end{pmatrix} = \mathbf{0} \qquad \text{implies} \qquad \omega(\mathbf{d}) \leq 1\,, \tag{2.7}$$

where the matrix $\mathbf{U}^{(2)}$ depends on $(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})$. Let $\tilde{\mathbf{d}} = (d_1 d_2, d_1 d_3, \ldots, d_{R-1} d_R)^T$. From the form of $\tilde{\mathbf{d}}$ it can be seen that $\tilde{\mathbf{d}} = \mathbf{0}$ implies $\omega(\mathbf{d}) \leq 1$. This shows that $\mathbf{U}^{(2)}$ having full column rank is sufficient for condition (2.7) to hold. This condition is easy to check.

In Theorem 4.4, we show how to obtain a matrix $\mathbf{U}^{(n-1)}$ from $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$ such that

$$\mathbf{U}^{(n-1)}\, \tilde{\mathbf{d}} = \mathbf{0} \qquad \text{implies} \qquad \omega(\mathbf{d}) \leq 1 \tag{2.8}$$

is equivalent to (2.6) for arbitrary $n \geq 3$. Moreover, in Corollary 4.5, we show that $\mathbf{U}^{(n-1)}$ having full column rank is sufficient for uniqueness of $\mathbf{A}^{(n)}$ and, hence, for uniqueness of the complete nth order decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$. This generalizes the easy-to-check condition for $n = 3$ of [15].

For $n = 3$ and $n = 4$, this sufficient uniqueness condition (i.e., $\mathbf{U}^{(n-1)}$ having full column rank) is obtained independently by De Lathauwer [6]. Moreover, for $n = 3$, [6] shows that for generic $(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})$ the matrix $\mathbf{U}^{(2)}$ has full column rank if

$$\frac{R(R-1)}{2} \;\leq\; \frac{I_1(I_1-1)I_2(I_2-1)}{4}\,. \tag{2.9}$$

Also, for $n = 4$, [6] shows that for generic $(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{A}^{(3)})$ the matrix $\mathbf{U}^{(3)}$ has full column rank if

$$\frac{R(R-1)}{2} \;\leq\; \frac{I_1 I_2 I_3\,(3\,I_1 I_2 I_3 - I_1 I_2 - I_1 I_3 - I_2 I_3 - I_1 - I_2 - I_3 + 3)}{8}\,. \tag{2.10}$$

We refer to these types of uniqueness conditions as generic uniqueness conditions. It was observed by Stegeman, Ten Berge, and De Lathauwer [35] that (2.9) is equivalent to $\mathbf{U}^{(2)}$ being a square or vertical matrix (after redundant rows have been deleted). The latter authors also give an alternative proof of the generic uniqueness condition (2.9).

Stegeman [34] shows that Kruskal's uniqueness condition (1.2) with $k_{\mathbf{A}^{(3)}} = R$ implies that $\mathbf{U}^{(2)}$ has full column rank. Hence, the latter condition is less restrictive. Moreover, for $k_{\mathbf{A}^{(3)}} = R$ [34] proves several Kruskal-type uniqueness conditions that are less restrictive than (1.2) but more restrictive than $\mathbf{U}^{(2)}$ having full column rank.

From his constructive proofs of the deterministic uniqueness conditions for $n = 3$ and $n = 4$, De Lathauwer [6] shows that the decomposition (1.1) can be obtained algebraically from a simultaneous matrix diagonalization.

In Theorem 5.5, we use our derivation of the matrix $\mathbf{U}^{(n-1)}$ to prove generic uniqueness conditions for arbitrary $n \geq 3$ that generalize conditions (2.9) and (2.10) of De Lathauwer [6]. Our approach is as follows. We identify rows of the matrix $\mathbf{U}^{(n-1)}$ that are redundant (i.e., can be deleted without affecting the row space) for any $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$. The matrix $\mathbf{U}^{(n-1)}$ has $R(R-1)/2$ columns, and it can only have full column rank if $R(R-1)/2$ is less than or equal to the number of nonredundant rows. For $n = 3$, this yields condition (2.9), as was observed by [35]. For $n = 4$, we show that this yields condition (2.10). For arbitrary $n \geq 3$, we prove an expression for the number of nonredundant rows of $\mathbf{U}^{(n-1)}$. Our generic uniqueness conditions state that, for generic $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$, the matrix $\mathbf{U}^{(n-1)}$ has full column rank if $R(R-1)/2$ is less than or equal to the number of nonredundant rows.

Our results provide easy-to-check uniqueness conditions for the decomposition (1.1) in case the vectors $\mathbf{a}_r^{(j)}$, $r = 1, \ldots, R$, are linearly independent in (at least) one mode j. Moreover, our proofs offer more insight into uniqueness of tensor decompositions into rank-1 terms for arbitrary order $n \geq 3$.

This paper is organized as follows. In section 3 we present generalizations of well-known necessary uniqueness conditions for $n = 3$. In section 4 we generalize the approach of Jiang and Sidiropoulos [15] to arbitrary order $n \geq 3$ and obtain our uniqueness conditions. In section 5, we generalize the generic uniqueness conditions of De Lathauwer [6] by identifying redundant rows of $\mathbf{U}^{(n-1)}$ for arbitrary $n \geq 3$. Finally, section 6 contains a discussion of our results.

Although we consider the real-valued nth order tensor decomposition, all presented uniqueness results can easily be translated to the complex-valued case. To do this, we must keep in mind that our vectors live in a complex vector space $\mathbb{C}^m$, with inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^H \mathbf{x}$ and norm $||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$, where ^{H} denotes the Hermitian or conjugated transpose. As in $\mathbb{R}^m$, vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal when $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Also, vectors $\mathbf{x}_1, \ldots, \mathbf{x}_q \in \mathbb{C}^m$ are linearly independent when $a_1 \mathbf{x}_1 + \cdots + a_q \mathbf{x}_q = \mathbf{0}$ implies $a_1 = \cdots = a_q = 0$ for scalars $a_1, \ldots, a_q \in \mathbb{C}$. Moreover, the determinant of a complex matrix is defined identical to the determinant of a real matrix, and its relation to the matrix rank is identical. The considerations above imply that, in order to translate our uniqueness proofs to the complex-valued case, we must replace the ordinary transpose ^{T} by ^{H} where orthogonality is involved; for example, see Lemma 2.1 and the discussion following it. However, in those cases where the transpose is due to the formulation of the decomposition such as in (2.1), (2.2), (2.3), the transpose should not be changed. See [27] for a proof of Kruskal's condition (1.2) for the complex case, and [15] for a proof of Kruskal's permutation lemma (Lemma 2.1) for the complex case. Moreover, all uniqueness results of [15] are proven for the complex case.

3. Necessary uniqueness conditions for the nth order decomposition.

Here, we present necessary uniqueness conditions for a decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$. These are obtained by generalizing necessary uniqueness conditions for $n = 3$, and serve to illustrate differences and similarities between the often studied case $n = 3$ and the case $n \geq 4$.

For $n = 3$, a necessary uniqueness condition is that the Khatri–Rao product of any two component matrices must have full column rank; see Liu and Sidiropoulos [23]. The next lemma generalizes this condition to an nth order decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$. Recall the definition of the mode-j matrix unfolding from (2.1).

LEMMA 3.1. *If* $\mathrm{rank}(\bigodot_{i \neq j}^n \mathbf{A}^{(i)}) < R$ *for some* $j \in \{1, \ldots, n\}$*,* $n \geq 3$*, then the decomposition* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ *is not unique up to permutation and scaling. Moreover, an alternative decomposition into* $R - 1$ *rank*-1 *terms exists.*

Proof. The proof is analogous to the proof in Stegeman and Sidiropoulos [36] for $n = 3$. Suppose $(\bigodot_{i \neq j}^n \mathbf{A}^{(i)})\, \mathbf{x} = \mathbf{0}$ for some nonzero vector $\mathbf{x}$. Then the mode-j

matrix unfolding of the decomposition satisfies

$$\left(\bigodot_{i\neq j}^{n} \mathbf{A}^{(i)}\right) (\mathbf{A}^{(j)})^T = \left(\bigodot_{i\neq j}^{n} \mathbf{A}^{(i)}\right) (\mathbf{A}^{(j)} + \mathbf{y}\mathbf{x}^T)^T \tag{3.1}$$

for any vector $\mathbf{y}$. Hence, in the decomposition we may replace $\mathbf{A}^{(j)}$ by $(\mathbf{A}^{(j)} + \mathbf{y}\mathbf{x}^T)$ for any vector $\mathbf{y}$. This proves nonuniqueness. Moreover, we can choose $\mathbf{y}$ such that one column, say column p, of $(\mathbf{A}^{(j)} + \mathbf{y}\mathbf{x}^T)$ vanishes. Hence, a decomposition into $R-1$ rank-1 terms can be obtained by deleting columns $\mathbf{a}_p^{(i)}$ from each component matrix $\mathbf{A}^{(i)}$, $i \neq j$, and replacing $\mathbf{A}^{(j)}$ by $(\mathbf{A}^{(j)} + \mathbf{y}\mathbf{x}^T)$ with its all-zero column p deleted. □

From Lemma 3.1 it is clear that an all-zero column in one of the component matrices (which thus has k-rank zero) implies nonuniqueness of the decomposition. For $n = 3$, it is well known that a component matrix with k-rank one (proportional columns) implies nonuniqueness; see, e.g., Stegeman and Sidiropoulos [36]. However, as observed by Sidiropoulos and Bro [26], for $n \geq 4$ component matrices may have k-rank one while the decomposition is unique. Indeed, suppose a 3rd order decomposition satisfies Kruskal's condition (1.2). Adding a fourth component matrix with k-rank one now yields a 4th order decomposition that satisfies the uniqueness condition (1.3) for $n = 4$.

The next lemma generalizes the necessary uniqueness condition for $n = 3$ of k-rank at least two to arbitrary order $n \geq 3$.

LEMMA 3.2. *If the decomposition* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$, $n \geq 3$, *contains* $n-2$ *distinct component matrices that have columns* s *and* t *proportional,* $s \neq t$, *then the decomposition is not unique up to permutation and scaling.*

Proof. The proof is analogous to the proof in Stegeman and Sidiropoulos [36] for $n = 3$. Let $\mathbf{a}_s^{(j)} = \alpha^{(j)} \mathbf{a}_t^{(j)}$ for $j = 1, \ldots, n-2$. For the rank-1 terms s and t of the decomposition we have

$$\mathbf{a}_s^{(1)} \circ \cdots \circ \mathbf{a}_s^{(n)} + \mathbf{a}_t^{(1)} \circ \cdots \circ \mathbf{a}_t^{(n)} = \mathbf{a}_t^{(1)} \circ \cdots \circ \mathbf{a}_t^{(n-2)} \circ [\tilde{\alpha}\, \mathbf{a}_s^{(n-1)} | \mathbf{a}_t^{(n-1)}]\, [\mathbf{a}_s^{(n)} | \mathbf{a}_t^{(n)}]^T$$

$$= \mathbf{a}_t^{(1)} \circ \cdots \circ \mathbf{a}_t^{(n-2)} \circ [\tilde{\alpha}\, \mathbf{a}_s^{(n-1)} | \mathbf{a}_t^{(n-1)}]\, \mathbf{U} ([\mathbf{a}_s^{(n)} | \mathbf{a}_t^{(n)}]\, \mathbf{U}^{-T})^T, \tag{3.2}$$

with $\tilde{\alpha} = \prod_{j=1}^{n-2} \alpha^{(j)}$, and $\mathbf{U}$ a nonsingular 2×2 matrix. If $\mathbf{U}$ is not the product of a permutation matrix and a nonsingular diagonal matrix, then (3.2) implies nonuniqueness. □

Note that in the proof of Lemma 3.2 the nonuniqueness of the matrix decomposition (2nd order) is used. Since the nth order decomposition is unique under mild conditions for $n \geq 3$, it is not possible to write an analogous proof of nonuniqueness for the case where less than $n-2$ distinct component matrices have columns s and t proportional.

4. Uniqueness conditions for the nth order decomposition.

Here, we present uniqueness conditions for a decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$ with $\mathrm{rank}(\mathbf{A}^{(n)}) = R$. We denote an alternative decomposition as $(\mathbf{B}^{(1)}, \ldots, \mathbf{B}^{(n)})$. It is assumed that the necessary uniqueness conditions of Lemmas 3.1 and 3.2 hold. Our approach is a generalization of Jiang and Sidiropoulos [15], and focuses on equating the mode-n matrix unfoldings of the two decompositions as in (2.2).

The next lemma shows that we need only prove uniqueness of $\mathbf{A}^{(n)}$ to obtain uniqueness of the complete decomposition $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$.

LEMMA 4.1. *Let* $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(n)})$, $n \geq 3$, *be a decomposition with* $\operatorname{rank}(\mathbf{A}^{(n)}) = R$. *If for any alternative decomposition* $(\mathbf{B}^{(1)},\ldots,\mathbf{B}^{(n)})$ *there holds* $\mathbf{B}^{(n)} = \mathbf{A}^{(n)}\mathbf{\Pi}\,\mathbf{\Lambda}_n$ *for a permutation matrix* $\mathbf{\Pi}$ *and a nonsingular diagonal matrix* $\mathbf{\Lambda}_n$, *then* $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(n)})$ *is unique up to permutation and scaling.*

Proof. The proof is analogous to Jiang and Sidiropoulos [15] for $n = 3$. As stated in section 2, under uniqueness of $\mathbf{A}^{(n)}$, (2.2) implies (2.4). Hence, each column of $(\mathbf{B}^{(1)}\odot\cdots\odot\mathbf{B}^{(n-1)})$ is a rescaled column of $(\mathbf{A}^{(1)}\odot\cdots\odot\mathbf{A}^{(n-1)})$. Each such column r can be interpreted as a vectorized $(n-1)$th order tensor that is the outer product of $n-1$ vectors that are the rth columns of the $n-1$ component matrices. Since the component matrices $\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(n-1)}$ do not contain all-zero columns by assumption (otherwise $\mathbf{A}^{(n)}$ would not be unique), it now follows that $\mathbf{B}^{(j)} = \mathbf{A}^{(j)}\mathbf{\Pi}\,\mathbf{\Lambda}_j$, $j = 1,\ldots,n-1$, for nonsingular diagonal matrices $\mathbf{\Lambda}_j$. Since $(\mathbf{A}^{(1)}\odot\cdots\odot\mathbf{A}^{(n-1)})$ has full column rank (otherwise $\mathbf{A}^{(n)}$ would not be unique; see Lemma 3.1), (2.4) implies that $\prod_{j=1}^{n-1}\mathbf{\Lambda}_j = \mathbf{\Lambda}_n^{-1}$. Hence, the decomposition $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(n)})$ is unique up to permutation and scaling. ∎

In the remaining part of this section, we focus on the uniqueness of $\mathbf{A}^{(n)}$. As explained in section 2, we use Kruskal's [21] permutation lemma, which is formulated as Lemma 2.1. In section 2, we derived the sufficient uniqueness condition (4.1). The next theorem shows that this condition is not only sufficient but also necessary for uniqueness. For $n = 3$, this result is due to Jiang and Sidiropoulos [15].

THEOREM 4.2. *Let* $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(n)})$, $n \geq 3$, *be a decomposition with* $\operatorname{rank}(\mathbf{A}^{(n)}) = R$. *Then the decomposition is unique up to permutation and scaling if and only if for any vector* $\mathbf{d}$

$$(\mathbf{A}^{(1)}\odot\cdots\odot\mathbf{A}^{(n-1)})\,\mathbf{d} = (\mathbf{f}_1\otimes\cdots\otimes\mathbf{f}_{n-1}) \qquad \textit{implies} \qquad \omega(\mathbf{d}) \leq 1\,. \tag{4.1}$$

Proof. Sufficiency follows from the analysis in section 2. Condition (4.1) implies uniqueness of $\mathbf{A}^{(n)}$ via the permutation lemma. Uniqueness of the complete decomposition follows from Lemma 4.1.

The proof of necessity is as follows. Without loss of generality we set $\mathbf{A}^{(n)} = \mathbf{I}_R$. Suppose $(\mathbf{A}^{(1)}\odot\cdots\odot\mathbf{A}^{(n-1)})\,\mathbf{d} = (\mathbf{f}_1\otimes\cdots\otimes\mathbf{f}_{n-1})$ for some vector $\mathbf{d}$ with $\omega(\mathbf{d}) \geq 2$. Let $d_p \neq 0$, and set $\mathbf{B}^{(j)}$ equal to $\mathbf{A}^{(j)}$ with column p replaced by $\mathbf{f}_j$, $j = 1,\ldots,n-1$. Then $(\mathbf{a}_p^{(1)}\otimes\cdots\otimes\mathbf{a}_p^{(n-1)}) = (\mathbf{B}^{(1)}\odot\cdots\odot\mathbf{B}^{(n-1)})\,\mathbf{g}$ for some vector $\mathbf{g}$ with $\omega(\mathbf{g}) \geq 2$. Let $\mathbf{B}^{(n)}$ be equal to $\mathbf{I}_R$ with row p replaced by $\mathbf{g}^T$. We have

$$(\mathbf{A}^{(1)}\odot\cdots\odot\mathbf{A}^{(n-1)}) = (\mathbf{B}^{(1)}\odot\cdots\odot\mathbf{B}^{(n-1)})\,(\mathbf{B}^{(n)})^T\,, \tag{4.2}$$

which shows that $(\mathbf{B}^{(1)},\ldots,\mathbf{B}^{(n)})$ is an alternative decomposition to $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(n-1)},\mathbf{I}_R)$. Since the alternative component matrices $\mathbf{B}^{(j)}$ are not rescaled column permutations of the original component matrices $\mathbf{A}^{(j)}$, this shows nonuniqueness of the decomposition. ∎

Condition (4.1) is difficult to check. Next, we prove an equivalent uniqueness condition that is a generalization of (2.7). The left-hand side of (4.1) can be interpreted as a vectorized $(n-1)$th order tensor $\underline{\mathbf{Y}}$ that is a linear combination (with coefficients in $\mathbf{d}$) of R rank-1 tensors specified by $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(n-1)})$. Hence,

$$\underline{\mathbf{Y}} = \sum_{r=1}^{R} d_r\,(\mathbf{a}_r^{(1)}\circ\cdots\circ\mathbf{a}_r^{(n-1)})\,. \tag{4.3}$$

The right-hand side of (4.1) can be interpreted as a vectorized $(n-1)$th order tensor of rank at most 1 (since it is the outer product of $n-1$ vectors). Condition (4.1) is equivalent to

$$\text{rank}(\underline{\mathbf{Y}}) \leq 1 \qquad \text{implies} \qquad \omega(\mathbf{d}) \leq 1\,. \tag{4.4}$$

Condition (4.4) states that if $\underline{\mathbf{Y}}$ has rank at most 1, then at most one coefficient in the linear combination (4.3) can be nonzero.

An mth order tensor has rank at most 1 if and only if its m matrix unfoldings have rank at most 1. This is proven in Lemma 4.6, which is postponed for ease of presentation. We apply this rank-1 criterion to $\underline{\mathbf{Y}}$ to obtain a matrix $\mathbf{U}^{(n-1)}$ that is a generalization of $\mathbf{U}^{(2)}$ in condition (2.7). For this, we need the following definition of a matrix containing all distinct 2×2 minors of a given matrix.

DEFINITION 4.3. *For an $I \times R$ matrix $\mathbf{A}$, let the $I(I-1)/2 \times R(R-1)/2$ matrix $m(\mathbf{A})$ have entries*

$$\det \begin{pmatrix} a_{ig} & a_{ih} \\ a_{jg} & a_{jh} \end{pmatrix}, \qquad \textit{with} \quad 1 \leq i < j \leq I \quad \textit{and} \quad 1 \leq g < h \leq R\,, \tag{4.5}$$

where in each row of $m(\mathbf{A})$ the value of (i,j) is fixed, and in each column of $m(\mathbf{A})$ the value of (g,h) is fixed. The columns of $m(\mathbf{A})$ are ordered such that index g runs slower than h. The rows of $m(\mathbf{A})$ are ordered such that index i runs slower than j. □

It is clear that $\text{rank}(\mathbf{A}) \leq 1$ is equivalent to $m(\mathbf{A}) = \mathbf{O}$.

The tensor $\underline{\mathbf{Y}}$ has mode-j matrix unfolding, $(\bigodot_{i \neq j}^{n-1} \mathbf{A}^{(i)})\,\text{diag}(\mathbf{d})\,(\mathbf{A}^{(j)})^T$, $j = 1, \ldots, n-1$; see (2.1). By Lemma 4.6, condition (4.4) is equivalent to

$$m\left(\left(\bigodot_{i \neq j}^{n-1} \mathbf{A}^{(i)}\right) \text{diag}(\mathbf{d})\,(\mathbf{A}^{(j)})^T\right) = \mathbf{O}, \quad j = 1, \ldots, n-1, \quad \text{implies} \quad \omega(\mathbf{d}) \leq 1. \tag{4.6}$$

For $n = 3$, Jiang and Sidiropoulos [15] show that $m(\mathbf{A}^{(1)}\,\text{diag}(\mathbf{d})\,(\mathbf{A}^{(2)})^T) = \mathbf{O}$ can be rewritten as $m(\mathbf{A}^{(1)}) \odot m(\mathbf{A}^{(2)})\,\tilde{\mathbf{d}} = \mathbf{0}$, where $\tilde{\mathbf{d}} = (d_1 d_2, d_1 d_3, \ldots, d_{R-1} d_R)^T$. Analogously, we obtain that $m((\bigodot_{i \neq j}^{n-1} \mathbf{A}^{(i)})\,\text{diag}(\mathbf{d})\,(\mathbf{A}^{(j)})^T) = \mathbf{O}$ can be rewritten as

$$m\left(\bigodot_{i \neq j}^{n-1} \mathbf{A}^{(i)}\right) \odot m\left(\mathbf{A}^{(j)}\right) \tilde{\mathbf{d}} = \mathbf{0}\,. \tag{4.7}$$

Note that each row of (4.7) corresponds to a distinct 2×2 minor of the mode-j matrix unfolding of $\underline{\mathbf{Y}}$. The system (4.7) contains all distinct 2×2 minors of this matrix unfolding and, hence, is equivalent to the matrix unfolding having rank at most 1. Next, we combine the equations (4.7), $j = 1, \ldots, n-1$, in one system of equations. Let

$$\mathbf{U}_j^{(n-1)} = m\left(\bigodot_{i \neq j}^{n-1} \mathbf{A}^{(i)}\right) \odot m\left(\mathbf{A}^{(j)}\right), \qquad j = 1, \ldots, n-1\,, \tag{4.8}$$

and define

$$\mathbf{U}^{(n-1)} = \begin{bmatrix} \mathbf{U}_1^{(n-1)} \\ \vdots \\ \mathbf{U}_{n-1}^{(n-1)} \end{bmatrix}. \tag{4.9}$$

This yields the following equivalent necessary and sufficient uniqueness condition, which is a generalization of (2.7).

THEOREM 4.4. *Let* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$*,* $n \geq 3$*, be a decomposition with* $\text{rank}(\mathbf{A}^{(n)}) = R$*. Then the decomposition is unique up to permutation and scaling if and only if for any vector* $\mathbf{d}$

$$\mathbf{U}^{(n-1)}\,\tilde{\mathbf{d}} = \mathbf{0} \qquad \textit{implies} \qquad \omega(\mathbf{d}) \leq 1\,. \tag{4.10}$$

Proof. The analysis above shows that $\mathbf{U}^{(n-1)}\,\tilde{\mathbf{d}} = \mathbf{0}$ is equivalent to all $n-1$ matrix unfoldings of $\underline{\mathbf{Y}}$ having rank at most 1. By Lemma 4.6, this is equivalent to $\text{rank}(\underline{\mathbf{Y}}) \leq 1$. Hence, condition (4.10) is equivalent to condition (4.4), which was shown to be equivalent to condition (4.1). Theorem 4.2 completes the proof. ∎

From the form of $\tilde{\mathbf{d}}$ it can be seen that $\tilde{\mathbf{d}} = \mathbf{0}$ implies $\omega(\mathbf{d}) \leq 1$. It follows that a sufficient condition for uniqueness is that $\mathbf{U}^{(n-1)}$ has full column rank, i.e., $\text{rank}(\mathbf{U}^{(n-1)}) = R(R-1)/2$. We formulate this as a corollary.

COROLLARY 4.5. *Let* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$*,* $n \geq 3$*, be a decomposition with* rank $(\mathbf{A}^{(n)}) = R$*. Then the decomposition is unique up to permutation and scaling if* $\mathbf{U}^{(n-1)}$ *has full column rank.* ∎

Theorems 4.2 and 4.4 and Corollary 4.5 are generalizations of the uniqueness conditions of Jiang and Sidiropoulos [15] for $n = 3$. Corollary 4.5 was independently proven by De Lathauwer [6] for $n = 3$ and $n = 4$. Note that $\mathbf{U}^{(n-1)}$ having full column rank is an easy-to-check uniqueness condition compared to condition (4.1).

It remains to formulate and prove Lemma 4.6.

LEMMA 4.6. *An* m*th order tensor* $\underline{\mathbf{X}}$ *has rank at most* 1 *if and only if its mode-*j *matrix unfolding has rank at most* 1*,* $j = 1, \ldots, m$*,* $m \geq 2$*.*

Proof. Suppose $\underline{\mathbf{X}}$ has rank at most 1. Then we have the representation

$$\underline{\mathbf{X}} = \mathbf{a}^{(1)} \circ \cdots \circ \mathbf{a}^{(m)} \tag{4.11}$$

for some vectors $\mathbf{a}^{(j)}$, $j = 1, \ldots, m$. The mode-j matrix unfolding of $\underline{\mathbf{X}}$ is given by $(\bigodot_{i \neq j}^{m} \mathbf{a}^{(i)})\,(\mathbf{a}^{(j)})^T$ (see (2.1)), which is the outer product of two vectors and, hence, has rank at most 1.

Next, suppose all m matrix unfoldings of $\underline{\mathbf{X}}$ have rank at most 1. This implies that all mode-j vectors of $\underline{\mathbf{X}}$ are proportional to some vector $\mathbf{a}^{(j)}$, $j = 1, \ldots, m$. Hence, tensor $\underline{\mathbf{X}}$ is defined on the outer product of m linear spaces $\mathcal{S}_\ell$, with $\dim(\mathcal{S}_\ell) \leq 1$, $1 \leq \ell \leq m$, and a representation (4.11) is possible. This shows that $\underline{\mathbf{X}}$ has at most rank 1. ∎

5. Generic uniqueness conditions for the nth order decomposition.

In Lemma 4.6, we do not need to check all distinct 2×2 minors of all matrix unfoldings of the tensor to conclude that it has rank at most 1. In this section, we identify the 2×2 minors that are redundant when checking that $\text{rank}(\underline{\mathbf{Y}}) \leq 1$ in condition (4.4). Since each 2×2 minor corresponds to a row in $\mathbf{U}^{(n-1)}$, a redundant minor corresponds to a redundant row of $\mathbf{U}^{(n-1)}$. We distinguish the following ways in which a row of $\mathbf{U}^{(n-1)}$ can be redundant (i.e., can be deleted without affecting the row space). Rows of $\mathbf{U}^{(n-1)}$ can be redundant due to the following:

- (I) the corresponding minor being redundant for any $\underline{\mathbf{Y}}$ (i.e., for any $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$, for any $I_1, \ldots, I_{n-1}$, for any $R \geq 2$, for any $\mathbf{d}$), or
- (II) the corresponding minor being redundant not due to (I), but due to the particular values of $I_1, \ldots, I_{n-1}$ and $R \geq 2$ at hand, for any $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$, for any $\mathbf{d}$, or
- (III) the corresponding minor being redundant not due to (I) or (II), but due to the particular $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$ at hand, for any $\mathbf{d}$.

For $R = 1$ the decomposition is always unique up to permutation and scaling (except if the tensor to be decomposed is all-zero). In this section, we assume $R \geq 2$.

We define $\tilde{\mathbf{U}}^{(n-1)}$ as the matrix that is obtained by deleting redundant rows of type (I) from $\mathbf{U}^{(n-1)}$. Redundant rows of type (II) occur when $\tilde{\mathbf{U}}^{(n-1)}$ has redundant rows due to the particular values of $I_1, \ldots, I_{n-1}$ and $R \geq 2$ at hand, for any $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$. This happens when $\tilde{\mathbf{U}}^{(n-1)}$ has more rows than columns. Redundant rows of type (III) occur when $\tilde{\mathbf{U}}^{(n-1)}$ is a square or horizontal matrix and does not have full row rank. In Corollary 4.5, full column rank of $\mathbf{U}^{(n-1)}$ implies uniqueness of the decomposition. Matrix $\mathbf{U}^{(n-1)}$ has full column rank if and only if $\tilde{\mathbf{U}}^{(n-1)}$ has full column rank. This implies that $\tilde{\mathbf{U}}^{(n-1)}$ may not have more columns than rows. Hence, the number of nonredundant minors of type (I) is a necessary upper bound on $R(R-1)/2$ for Corollary 4.5 to hold. We show that, for $n = 3$ and $n = 4$, these upper bounds are identical to the generic uniqueness bounds (2.9)–(2.10) obtained by De Lathauwer [6]. Moreover, we show that analogous generic uniqueness bounds can be obtained for arbitrary order $n \geq 3$. For generic $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$, redundant minors of type (III) do not occur by definition, and $\tilde{\mathbf{U}}^{(n-1)}$ has full column rank if it is a square or vertical matrix. These observations also underly the proofs of the generic uniqueness bounds (2.9)–(2.10) in [6].

In section 5.1 we present our analysis of redundant minors of type (I) for $n \geq 3$. In section 5.2 we prove that our approach yields generic uniqueness bounds, and illustrate our result by computing the bounds on $R(R-1)/2$ for $n = 3$, $n = 4$, and $n = 5$. In section 5.3 some numerical examples are given.

5.1. Identifying redundant 2×2 minors of type (I). A 2×2 minor of the mode-j matrix unfolding of the $(n-1)$th order tensor $\underline{\mathbf{Y}}$ in (4.3) corresponds to an equation $y_{\mathcal{A}}\, y_{\mathcal{D}} = y_{\mathcal{B}}\, y_{\mathcal{C}}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ contain $n-1$ indices. In the following, we will refer to this equation as a minor as well. The mode-j matrix unfolding contains mode-j vectors as columns. We assume (without loss of generality) that $y_{\mathcal{A}}$ and $y_{\mathcal{C}}$ are entries of the same mode-j vector and, hence, have all indices except the jth index identical. Let the $n-2$ identical indices be contained in $\mathcal{I}_1$. Analogously, $y_{\mathcal{B}}$ and $y_{\mathcal{D}}$ are entries of an other mode-j vector defined by $n-2$ indices in $\mathcal{I}_2$. Let $\text{dif}(\mathcal{A}, \mathcal{D})$ denote the number of different indices in $\mathcal{A}$ and $\mathcal{D}$ (i.e., indices with different values at the same position). It follows that $\text{dif}(\mathcal{A}, \mathcal{D}) = 1 + \text{dif}(\mathcal{I}_1, \mathcal{I}_2) \in \{2, \ldots, n-1\}$. Note that $\text{dif}(\mathcal{A}, \mathcal{D}) = \text{dif}(\mathcal{D}, \mathcal{A})$, and $\text{dif}(\mathcal{A}, \mathcal{D}) = \text{dif}(\mathcal{B}, \mathcal{C})$.

It is our goal to identify redundant 2×2 minors of type (I) among all distinct minors of all $n-1$ matrix unfoldings of an $I_1 \times \cdots \times I_{n-1}$ tensor $\underline{\mathbf{Y}}$ when checking that $\text{rank}(\underline{\mathbf{Y}}) \leq 1$. Since we consider redundant minors of type (I), we do not assume knowledge of which entries of $\underline{\mathbf{Y}}$ are nonzero. We only identify minors that are redundant for all possible decompositions in (4.3). Therefore, in order to identify redundant minors we need only consider identical terms in the equations corresponding to the minors. Here, each term is the product of two entries of $\underline{\mathbf{Y}}$ as in $y_{\mathcal{A}}\, y_{\mathcal{D}} = y_{\mathcal{B}}\, y_{\mathcal{C}}$.

We obtain the number of redundant minors of type (I) as follows. First, we partition the minors into subsets such that minors in different subsets do not have identical terms. Next, the number of redundant minors of type (I) in each subset is identified. Finally, the total number of redundant minors of type (I) is obtained by adding the numbers of redundant minors in each subset. We begin this procedure by defining the order of a minor.

DEFINITION 5.1. *A 2×2 minor corresponding to an equation $y_{\mathcal{A}}\, y_{\mathcal{D}} = y_{\mathcal{B}}\, y_{\mathcal{C}}$ has* order $m = \text{dif}(\mathcal{A}, \mathcal{D})$. ∎

A minor $y_{\mathcal{A}}\, y_{\mathcal{D}} = y_{\mathcal{B}}\, y_{\mathcal{C}}$ of order m is a minor of a matrix unfolding of an mth order $2\times\cdots\times 2$ subtensor of $\underline{\mathbf{Y}}$, where the subtensor is completely defined by $(\mathcal{A},\mathcal{D})$. Indeed, for each index not identical in $\mathcal{A}$ and $\mathcal{D}$ the order of the subtensor increases by one, and the indices of all entries of the subtensor are known when $(\mathcal{A},\mathcal{D})$ is known. More formally, let $\mathcal{A} = \{i_1,\ldots,i_{n-1}\}$ and $\mathcal{D} = \{j_1,\ldots,j_{n-1}\}$. The mth order subtensor then contains the entries $y_{(l_1,\ldots,l_{n-1})}$ with $l_k = i_k$ if $i_k = j_k$ and $l_k \in \{i_k, j_k\}$ if $i_k \neq j_k$. The number of indices with $i_k \neq j_k$ is equal to $m = \mathrm{dif}(\mathcal{A},\mathcal{D})$. We denote the set of indices of the entries of the subtensor defined by $(\mathcal{A},\mathcal{D})$ as $\mathrm{subind}(\mathcal{A},\mathcal{D})$. Note that $\mathrm{subind}(\mathcal{A},\mathcal{D}) = \mathrm{subind}(\mathcal{D},\mathcal{A})$ and $\mathrm{subind}(\mathcal{A},\mathcal{D}) = \mathrm{subind}(\mathcal{B},\mathcal{C})$.

DEFINITION 5.2. *A 2×2 minor corresponding to an equation $y_{\mathcal{A}}\, y_{\mathcal{D}} = y_{\mathcal{B}}\, y_{\mathcal{C}}$ defines a $2\times\cdots\times 2$ subtensor of $\underline{\mathbf{Y}}$ with indices of entries in* $\mathrm{subind}(\mathcal{A},\mathcal{D})$. *The order of the subtensor equals $m = \mathrm{dif}(\mathcal{A},\mathcal{D})$.* ∎

The next lemma shows that minors that have different orders do not have identical terms, nor do minors of the same order but corresponding to different subtensors.

LEMMA 5.3. *Let $y_{\mathcal{A}}\, y_{\mathcal{D}} = y_{\mathcal{B}}\, y_{\mathcal{C}}$ and $y_{\tilde{\mathcal{A}}}\, y_{\tilde{\mathcal{D}}} = y_{\tilde{\mathcal{B}}}\, y_{\tilde{\mathcal{C}}}$ be two 2×2 minors of matrix unfoldings of an $(n-1)$th order tensor $\underline{\mathbf{Y}}$. Let $m = \mathrm{dif}(\mathcal{A},\mathcal{D})$ and $\tilde{m} = \mathrm{dif}(\tilde{\mathcal{A}},\tilde{\mathcal{D}})$. The minors do not have identical terms if $m \neq \tilde{m}$, or if $m = \tilde{m}$ and* $\mathrm{subind}(\mathcal{A},\mathcal{D}) \neq \mathrm{subind}(\tilde{\mathcal{A}},\tilde{\mathcal{D}})$.

Proof. Having identical terms $y_{\mathcal{A}}\, y_{\mathcal{D}} = y_{\tilde{\mathcal{A}}}\, y_{\tilde{\mathcal{D}}}$ implies identical orders $m = \tilde{m}$. Since $\mathrm{dif}(\mathcal{A},\mathcal{D}) = \mathrm{dif}(\mathcal{B},\mathcal{C})$ and $\mathrm{dif}(\tilde{\mathcal{A}},\tilde{\mathcal{D}}) = \mathrm{dif}(\tilde{\mathcal{B}},\tilde{\mathcal{C}})$, it follows that the minors do not have identical terms if $m \neq \tilde{m}$.

Next, suppose $m = \tilde{m}$ and $\mathrm{subind}(\mathcal{A},\mathcal{D}) \neq \mathrm{subind}(\tilde{\mathcal{A}},\tilde{\mathcal{D}})$. Having identical terms $y_{\mathcal{A}}\, y_{\mathcal{D}} = y_{\tilde{\mathcal{A}}}\, y_{\tilde{\mathcal{D}}}$ implies identical subtensors, i.e., $\mathrm{subind}(\mathcal{A},\mathcal{D}) = \mathrm{subind}(\tilde{\mathcal{A}},\tilde{\mathcal{D}})$. Since $\mathrm{subind}(\mathcal{A},\mathcal{D}) = \mathrm{subind}(\mathcal{B},\mathcal{C})$ and $\mathrm{subind}(\tilde{\mathcal{A}},\tilde{\mathcal{D}}) = \mathrm{subind}(\tilde{\mathcal{B}},\tilde{\mathcal{C}})$, it follows that the minors do not have identical terms if they correspond to different subtensors. ∎

It follows from Lemma 5.3 that we may identify minors with identical terms for each order and each subtensor separately. The next lemma identifies the number of redundant minors of type (I) in each subtensor for each order.

LEMMA 5.4. *Let $2 \leq m \leq n-1$. The number of mth order 2×2 minors of matrix unfoldings of an $(n-1)$th order tensor $\underline{\mathbf{Y}}$, that correspond to the same mth order $2\times\cdots\times 2$ subtensor of $\underline{\mathbf{Y}}$, equals $m\,2^{m-2}$. These minors contain 2^{m-1} distinct terms. The number of these minors that are type-*(I) *redundant equals $m\,2^{m-2} - 2^{m-1} + 1$. The number of type-*(I) *nonredundant minors equals $2^{m-1} - 1$.*

Proof. The number of distinct pairs $(\mathcal{A},\mathcal{D})$ with $m = \mathrm{dif}(\mathcal{A},\mathcal{D})$, which define the same mth order $2\times\cdots\times 2$ subtensor, equals 2^m. Here, each $(\mathcal{A},\mathcal{D})$ corresponds to some mth order minor of a matrix unfolding of the subtensor. Since terms $y_{\mathcal{A}}\, y_{\mathcal{D}}$ and $y_{\mathcal{D}}\, y_{\mathcal{A}}$ are considered identical, the number of distinct terms contained by the mth order minors equals 2^{m-1}. Each minor equates two of these terms. The $2\times 2^{m-1}$ mode-j matrix unfolding of the subtensor has the mode-j vectors as columns. Each mode-j vector has two entries and corresponds to $m-1$ fixed indices. An mth order minor is obtained from two mode-j vectors with no identical fixed indices. It follows that each matrix unfolding yields 2^{m-2} mth order minors in which each term appears exactly once. Hence, the total number of minors equals $m\,2^{m-2}$.

Consider two distinct terms $y_{\mathcal{A}}\, y_{\mathcal{D}}$ and $y_{\mathcal{B}}\, y_{\mathcal{C}}$ with $m = \mathrm{dif}(\mathcal{A},\mathcal{D}) = \mathrm{dif}(\mathcal{B},\mathcal{C})$. Let $\mathcal{A} = \{i_1,\ldots,i_{n-1}\}$ and $\mathcal{D} = \{j_1,\ldots,j_{n-1}\}$. For mth order minors $y_{\mathcal{A}}\, y_{\mathcal{D}} = y_{\mathcal{B}}\, y_{\mathcal{C}}$ that are obtained from the matrix unfoldings of the subtensor, the indices in $\mathcal{B}$ and $\mathcal{C}$ are obtained from $\mathcal{A}$ and $\mathcal{D}$ by swapping the indices in one pair (i_k, j_k) with $i_k \neq j_k$. Each matrix unfolding yields an mth order minor $y_{\mathcal{A}}\, y_{\mathcal{D}} = y_{\mathcal{B}}\, y_{\mathcal{C}}$ in which a different pair of indices is swapped. We claim that this implies that $y_{\mathcal{A}}\, y_{\mathcal{D}} = y_{\mathcal{B}}\, y_{\mathcal{C}}$ for any $(\mathcal{B},\mathcal{C})$

with $m = \text{dif}(\mathcal{B},\mathcal{C})$. Indeed, a chain can be constructed of terms $y_{\mathcal{B}_s}\, y_{\mathcal{C}_s}$, $s = 1, 2, \ldots$, that are equal to $y_{\mathcal{A}}\, y_{\mathcal{D}}$, such that in each step s one pair of indices is swapped in $\mathcal{A}$ and $\mathcal{D}$. The number of steps needed to get to $y_{\mathcal{B}}\, y_{\mathcal{C}}$ is at most $m-1$ (using m steps corresponds to swapping $\mathcal{A}$ and $\mathcal{D}$ entirely, which yields an identical term).

It follows from the above that the complete set of mth order minors is equivalent to all distinct terms $y_{\mathcal{A}}\, y_{\mathcal{D}}$ with $m = \text{dif}(\mathcal{A},\mathcal{D})$ being equal. However, since there are 2^{m-1} distinct terms, we need only $2^{m-1}-1$ minors equating two terms to have all terms equal. These are the type-(I) nonredundant minors. This completes the proof. ◻

For an mth order $2 \times \cdots \times 2$ subtensor, the corresponding mth order minors can be represented by a graph, where each node represents a distinct term and each edge connecting two nodes represents a minor equating the terms represented by the two nodes. In Figure 1, the graphs for $m = 2, 3, 4$ are depicted. For each m, the graph is connected (i.e., all nodes can be reached by traveling along the edges), which implies the equality of all terms. We can delete edges (i.e., redundant minors of type (I)) one by one such that the graph remains connected. The minimal number of edges needed (i.e., the number of type-(I) nonredundant minors) for this is equal to the number of nodes minus one (i.e., the number of terms minus one).

By Lemmas 5.3 and 5.4, the total number of redundant minors of type (I) is equal to

$$\sum_{m=2}^{n-1} (m\, 2^{m-2} - 2^{m-1} + 1)\, Q_{(m,n)}\,, \tag{5.1}$$

where $Q_{(m,n)}$ denotes the number of distinct mth order $2 \times \cdots \times 2$ subtensors of $\underline{\mathbf{Y}}$. Analogously, the total number of type-(I) nonredundant minors is equal to

$$\sum_{m=2}^{n-1} (2^{m-1} - 1)\, Q_{(m,n)}\,. \tag{5.2}$$

Note that half of the minors of 2nd and 3rd order are type-(I) redundant, since $m\, 2^{m-2} - 2^{m-1} + 1 = 2^{m-1} - 1$ for $m = 2, 3$. For $m \geq 4$, the number of type-(I) redundant mth order minors is larger than half of the total number of minors.

The numbers $Q_{(m,n)}$ are given by

$$Q_{(m,n)} = \sum_{\mathcal{S}_m} \prod_{j \in \mathcal{S}_m} \frac{I_j(I_j - 1)}{2} \prod_{j \notin \mathcal{S}_m} I_j\,, \tag{5.3}$$

where the summation is over all subsets $\mathcal{S}_m$ of $\{1, \ldots, n-1\}$ containing m distinct elements. If $m = n-1$, then we set $\prod_{j \notin \mathcal{S}_m} I_j = 1$.

5.2. Generic uniqueness bounds. The next theorem shows that our approach of identifying type-(I) redundant 2×2 minors yields generic uniqueness bounds for arbitrary $n \geq 3$.

THEOREM 5.5. *Let* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n)})$, $n \geq 3$, *be a decomposition with generic* $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$ *and* $\text{rank}(\mathbf{A}^{(n)}) = R$. *Then* $\mathbf{U}^{(n-1)}$ *has full column rank if*

$$\frac{R(R-1)}{2} \;\leq\; \sum_{m=2}^{n-1} (2^{m-1} - 1)\, Q_{(m,n)}\,. \tag{5.4}$$

Hence, the decomposition is unique up to permutation and scaling if (5.4) *holds.*

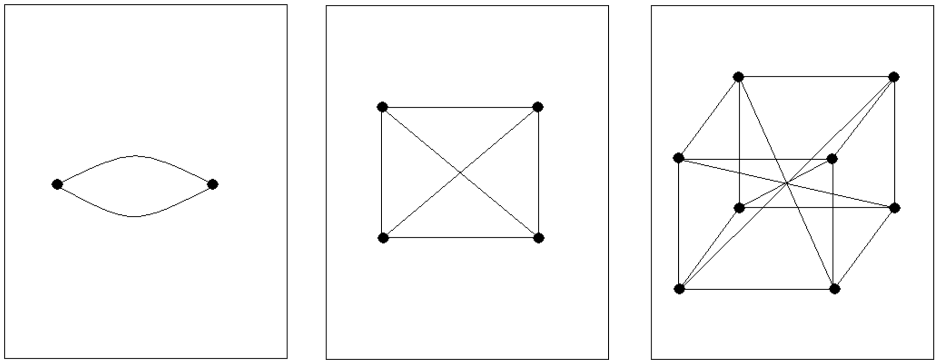

FIG. 1. *Graph representations of all mth order minors corresponding to the same mth order $2 \times \cdots \times 2$ subtensor, for $m = 2$ (left), $m = 3$ (middle), and $m = 4$ (right).*

Proof. For $R = 1$ the decomposition is always unique up to permutation and scaling (except if the tensor to be decomposed is all-zero). In the following, we assume $R \geq 2$.

Consider the tensor $\underline{\mathbf{Y}}$ in (4.3). Each row of $\mathbf{U}^{(n-1)}$ corresponds to a 2×2 minor of a matrix unfolding of $\underline{\mathbf{Y}}$. Recall the classification of redundant minors/rows at the beginning of this section. After deleting redundant rows of type (I) from $\mathbf{U}^{(n-1)}$, we obtain $\tilde{\mathbf{U}}^{(n-1)}$. For generic $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$, redundant rows of type (III) do not occur by definition. The nonredundant minors of type (I) can be written as a linear system $\tilde{\mathbf{U}}^{(n-1)}\tilde{\mathbf{d}}$. Redundant rows of type (II) are due to the fact that $\tilde{\mathbf{U}}^{(n-1)}$ has more rows than columns. For generic $(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)})$, this implies that $\tilde{\mathbf{U}}^{(n-1)}$ has full column rank if it is a square or vertical matrix. The matrix $\tilde{\mathbf{U}}^{(n-1)}$ has $R(R-1)/2$ columns and its number of rows is given by the right-hand side of (5.4). Hence, (5.4) implies that $\tilde{\mathbf{U}}^{(n-1)}$ generically has full column rank. Since $\tilde{\mathbf{U}}^{(n-1)}$ is obtained from $\mathbf{U}^{(n-1)}$ by deleting redundant rows, Corollary 4.5 completes the proof. ◻

Next, we compute and illustrate the generic uniqueness bound (5.4) for $n = 3, 4, 5$ and show that the bounds for $n = 3, 4$ coincide with the generic uniqueness bounds (2.9)–(2.10) of [6].

The bound for $n = 3$. Here, $\underline{\mathbf{Y}}$ is an $I_1 \times I_2$ matrix $\mathbf{Y}$. The $n - 1 = 2$ matrix unfoldings of $\mathbf{Y}$ are $\mathbf{Y}$ itself and $\mathbf{Y}^T$. Each 2×2 minor of $\mathbf{Y}$ has order 2 and is identical to a 2×2 minor of $\mathbf{Y}^T$. Hence, only the distinct 2×2 minors of $\mathbf{Y}$ are type-(I) nonredundant. Their number is given by

$$Q_{(2,3)} = \frac{I_1(I_1 - 1)I_2(I_2 - 1)}{4}\,, \tag{5.5}$$

which equals the right-hand side of (5.4) and is identical to the generic uniqueness bound (2.9) of [6]. The graph representing the two identical 2nd order minors of a 2×2 subtensor is depicted in the left-hand panel of Figure 1.

The bound for $n = 4$. Here, $\underline{\mathbf{Y}}$ is an $I_1 \times I_2 \times I_3$ tensor. We denote the $I_1 \times I_2$ frontal slices of $\underline{\mathbf{Y}}$ as $\mathbf{Y}_k$, $k = 1, \ldots, I_3$. The first and second matrix unfoldings are given by $[\mathbf{Y}_1|, \ldots, |\mathbf{Y}_{I_3}]$ and $[\mathbf{Y}_1^T|, \ldots, |\mathbf{Y}_{I_3}^T]$, respectively. Any 2×2 minor of $\mathbf{Y}_k$ has order 2 and is identical to a 2×2 minor of $\mathbf{Y}_k^T$, which yields

$$I_3 \, \frac{I_1(I_1 - 1)}{2} \, \frac{I_2(I_2 - 1)}{2} \tag{5.6}$$

identical minors. Analogous cases of identical minors occur between the first and third and between the second and third matrix unfoldings of $\underline{\mathbf{Y}}$. The total number of these identical minors is equal to the number of type-(I) (non)redundant minors of order 2. This number is given by the first term of (5.2), which equals

$$Q_{(2,4)} = I_1\,\frac{I_2(I_2-1)}{2}\,\frac{I_3(I_3-1)}{2} + I_2\,\frac{I_1(I_1-1)}{2}\,\frac{I_3(I_3-1)}{2} + I_3\,\frac{I_1(I_1-1)}{2}\,\frac{I_2(I_2-1)}{2}\,. \tag{5.7}$$

Next, we identify type-(I) redundant minors of order 3. We consider an arbitrary $2 \times 2 \times 2$ subtensor of $\underline{\mathbf{Y}}$ with frontal slices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e & f \\ g & h \end{bmatrix}. \tag{5.8}$$

The 3rd order minors corresponding to this subtensor are as follows:

$$ah = cf\,, \qquad bg = de \qquad (\text{matrix unfolding } 1)\,, \tag{5.9}$$

$$ah = bg\,, \qquad cf = de \qquad (\text{matrix unfolding } 2)\,, \tag{5.10}$$

$$ah = de\,, \qquad bg = cf \qquad (\text{matrix unfolding } 3)\,. \tag{5.11}$$

There are four distinct terms in (5.9)–(5.11) and they should all be equal. Only three of the six equations are needed for this. The graph representing the six minors of order 3 of a $2 \times 2 \times 2$ subtensor is depicted in the middle panel of Figure 1. The total number of type-(I) (non)redundant minors of order 3 equals the second term of (5.2), which is equal to

$$3\,Q_{(3,4)} \;=\; 3\,\frac{I_1(I_1-1)}{2}\,\frac{I_2(I_2-1)}{2}\,\frac{I_3(I_3-1)}{2}\,. \tag{5.12}$$

The total number of type-(I) (non)redundant minors equals the sum of (5.7) and (5.12), and can be rewritten as

$$\frac{I_1(I_1-1)}{4}\,\frac{I_2I_3(I_2I_3-1)}{2} + \frac{I_2(I_2-1)}{4}\,\frac{I_1I_3(I_1I_3-1)}{2} + \frac{I_3(I_3-1)}{4}\,\frac{I_1I_2(I_1I_2-1)}{2}\,, \tag{5.13}$$

which is half of all distinct 2×2 minors of the three matrix unfoldings of $\underline{\mathbf{Y}}$, and equals the right-hand side of (5.4). Moreover, (5.13) can be rewritten as the generic uniqueness bound (2.10) of [6]. Setting $I_3 = 1$ yields the bound for $n = 3$.

The bound for $n = 5$. Here, $\underline{\mathbf{Y}}$ is an $I_1 \times I_2 \times I_3 \times I_4$ tensor and has $n - 1 = 4$ matrix unfoldings. Using the analyses for $n = 3$ and $n = 4$, we can immediately see the number of type-(I) (non)redundant minors of orders 2 and 3. Indeed, analogous to (5.7) the number of type-(I) (non)redundant minors of order 2 is given by

$$\begin{aligned} Q_{(2,5)} = I_1I_2\,&\frac{I_3(I_3-1)}{2}\,\frac{I_4(I_4-1)}{2} + I_1I_3\,\frac{I_2(I_2-1)}{2}\,\frac{I_4(I_4-1)}{2} \\ &+ I_1I_4\,\frac{I_2(I_2-1)}{2}\,\frac{I_3(I_3-1)}{2} I_2I_3\,\frac{I_1(I_1-1)}{2}\,\frac{I_4(I_4-1)}{2} \\ &+ I_2I_4\,\frac{I_1(I_1-1)}{2}\,\frac{I_3(I_3-1)}{2} + I_3I_4\,\frac{I_1(I_1-1)}{2}\,\frac{I_2(I_2-1)}{2}. \end{aligned} \tag{5.14}$$

Analogous to (5.12), the number of type-(I) (non)redundant minors of order 3 is given by

$$3\,Q_{(3,5)} = 3\,I_1\ \frac{I_2(I_2-1)}{2}\,\frac{I_3(I_3-1)}{2}\,\frac{I_4(I_4-1)}{2} + 3\,I_2\ \frac{I_1(I_1-1)}{2}\,\frac{I_3(I_3-1)}{2}\,\frac{I_4(I_4-1)}{2} + 3\,I_3\ \frac{I_1(I_1-1)}{2}\,\frac{I_2(I_2-1)}{2}\,\frac{I_4(I_4-1)}{2} + 3\,I_4\ \frac{I_1(I_1-1)}{2}\,\frac{I_2(I_2-1)}{2}\,\frac{I_3(I_3-1)}{2}. \tag{5.15}$$

It remains to identify type-(I) redundant minors of order 4. Let an arbitrary $2\times2\times2\times2$ subtensor have frontal $2\times2\times2$ tensors

$$\left[\begin{array}{cc|cc} a & b & e & f \\ c & d & g & h \end{array}\right] \qquad \text{and} \qquad \left[\begin{array}{cc|cc} i & j & m & q \\ k & l & o & p \end{array}\right]. \tag{5.16}$$

The 4th order minors corresponding to (5.16) are as follows:

$$ap = cq, \quad bo = dm, \quad el = jg, \quad fk = hi \qquad \text{(matrix unfolding 1)}, \tag{5.17}$$

$$ap = bo, \quad cq = dm, \quad el = fk, \quad jg = hi \qquad \text{(matrix unfolding 2)}, \tag{5.18}$$

$$ap = el, \quad bo = fk, \quad cq = jg, \quad dm = hi \qquad \text{(matrix unfolding 3)}, \tag{5.19}$$

$$ap = hi, \quad bo = jg, \quad cq = fk, \quad dm = el \qquad \text{(matrix unfolding 4)}. \tag{5.20}$$

There are eight distinct terms in (5.17)–(5.20) and they should all be equal. It can be seen that only 7 of the 16 minors are needed for this. The graph representing the 16 minors of order 4 of a $2\times2\times2\times2$ subtensor is depicted in the right-hand panel of Figure 1. The total number of type-(I) redundant minors of order 4 equals

$$9\,Q_{(4,5)} = 9\ \frac{I_1(I_1-1)}{2}\,\frac{I_2(I_2-1)}{2}\,\frac{I_3(I_3-1)}{2}\,\frac{I_4(I_4-1)}{2}. \tag{5.21}$$

The total number of type-(I) redundant 2×2 minors equals the sum of (5.14), (5.15), and (5.21). Since for each $2\times2\times2\times2$ subtensor we have obtained 9 type-(I) redundant minors of 16, the total number of type-(I) redundant minors is larger than half of all distinct 2×2 minors of the four matrix unfoldings of $\underline{\mathbf{Y}}$. The generic uniqueness bound (5.4) is formed by the total number of type-(I) nonredundant minors. The total number of minors is given by

$$\frac{I_1(I_1-1)}{2}\,\frac{I_2I_3I_4(I_2I_3I_4-1)}{2} + \frac{I_2(I_2-1)}{2}\,\frac{I_1I_3I_4(I_1I_3I_4-1)}{2} + \frac{I_3(I_3-1)}{2}\,\frac{I_1I_2I_4(I_1I_2I_4-1)}{2} + \frac{I_4(I_4-1)}{2}\,\frac{I_1I_2I_3(I_1I_2I_3-1)}{2}. \tag{5.22}$$

Subtracting the total number of type-(I) redundant minors from (5.22) yields the uniqueness bound (5.4), which can be written as

$$\frac{R(R-1)}{2} \le \frac{I_1I_2I_3I_4}{16}\,(7\,I_1I_2I_3I_4 - I_1I_2I_3 - I_1I_2I_4 - I_1I_3I_4 - I_2I_3I_4 - I_1I_2 - I_1I_3 - I_1I_4 - I_2I_3 - I_2I_4 - I_3I_4 - I_1 - I_2 - I_3 - I_4 + 7)\,. \tag{5.23}$$

Setting $I_4 = 1$ yields the bound for $n = 4$. Setting $I_3 = I_4 = 1$ yields the bound for $n = 3$.

5.3. Numerical examples. We have programmed a MATLAB file that, for particular $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(n-1)})$, constructs the matrix $\mathbf{U}^{(n-1)}$ and computes the bound on $R(R-1)/2$ (right-hand side of (5.4)). For randomly sampled entries of $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(n-1)})$ from the standard Gaussian distribution, and various values of R, n, $I_1,\ldots,I_{n-1}$, we construct $\mathbf{U}^{(n-1)}$, determine its rank, and compare it with the bound on $R(R-1)/2$.

For $n = 4$, $I_1 = 2$, $I_2 = 3$, $I_3 = 4$, and $R = 16$, we have equality in (5.4). Indeed, both the left-hand side and right-hand side are equal to 120. The matrix $\mathbf{U}^{(3)}$ has 240 rows and 120 columns of which 120 rows are redundant. For random $(\mathbf{A}^{(1)},\mathbf{A}^{(2)},\mathbf{A}^{(3)})$, we get $\text{rank}(\mathbf{U}^{(3)}) = 120$ indeed.

Next, we set $n = 5$, $I_1 = 2$, $I_2 = 3$, $I_3 = 4$, and $I_4 = 2$. The bound (5.4) on $R(R-1)/2$ equals 636. This implies that R can be 36 at most. For random $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(4)})$ and $R = 36$, we get $\mathbf{U}^{(4)}$ with 1308 rows and 630 columns. The rank of $\mathbf{U}^{(4)}$ is 630. For $R = 37$, the matrix $\mathbf{U}^{(4)}$ has 666 columns, but its rank equals 636. Hence, $\mathbf{U}^{(4)}$ does not have full column rank in this case.

Finally, we set $n = 6$, $I_1 = 2$, $I_2 = 3$, $I_3 = 2$, $I_4 = 2$, and $I_5 = 3$. The bound (5.4) on $R(R-1)/2$ equals 1656, which implies that R can be 58 at most. For random $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(5)})$ and $R = 58$, we get $\mathbf{U}^{(5)}$ with 3546 rows and 1653 columns. The rank of $\mathbf{U}^{(5)}$ is 1653. For $R = 59$, the matrix $\mathbf{U}^{(5)}$ has 1711 columns, but its rank equals 1656.

For comparison, in the three examples above the largest values of R satisfying the generalization (1.3) of Kruskal's uniqueness condition (with $k_{\mathbf{A}^{(n)}} = R$ and $k_{\mathbf{A}^{(j)}} = \min(I_j,R) = I_j$, $j \leq n-1$) are 6, 7, and 7, respectively. This illustrates the large improvement of the generic uniqueness bound (5.4) with respect to (1.3). In Table 1, the examples are summarized.

Table 1
Examples of two generic uniqueness bounds on R for random decompositions $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(n)})$ with $\text{rank}(\mathbf{A}^{(n)}) = R$.

n	Size tensor	Bound on R from (1.3)	Bound on R from (5.4)
$n=4$	$2\times 3\times 4\times I_4,\ \ I_4 \geq R$	$R \leq 6$	$R \leq 16$
$n=5$	$2\times 3\times 4\times 2\times I_5,\ \ I_5 \geq R$	$R \leq 7$	$R \leq 36$
$n=6$	$2\times 3\times 2\times 2\times 3\times I_6,\ \ I_6 \geq R$	$R \leq 7$	$R \leq 58$

6. Discussion. In this paper, we have generalized various existing uniqueness conditions for the 3rd and 4th order decomposition to the nth order decomposition $(\mathbf{A}^{(1)},\ldots,\mathbf{A}^{(n)})$, for arbitrary $n \geq 3$. All uniqueness conditions assume that (at least) one of the component matrices $\mathbf{A}^{(j)}$ has rank R. The sufficient uniqueness condition of Corollary 4.5 and the generic uniqueness condition of Theorem 5.5 are easy to check.

The examples in section 5.3 illustrate the large improvement of the generic uniqueness condition with respect to the generalization (1.3) of Kruskal's uniqueness condition. Stegeman [34] has shown that for $n = 3$ and $k_{\mathbf{A}^{(n)}} = R$, Kruskal's uniqueness condition (1.2) implies that $\mathbf{U}^{(2)}$ has full column rank. Hence, for $n = 3$ the latter condition is less restrictive for any $(\mathbf{A}^{(1)},\mathbf{A}^{(2)})$. We conjecture that for $n \geq 3$, the same is true for the generalization (1.3) of Kruskal's condition and $\mathbf{U}^{(n-1)}$ having full column rank.

Stegeman, Ten Berge, and De Lathauwer [35] also consider the decomposition $(\mathbf{A}^{(1)}, \mathbf{A}^{(1)}, \mathbf{A}^{(3)})$ with symmetric slices and $\mathbf{A}^{(3)}$ having rank R. Analogous to Corollary 4.5, the decomposition is unique if $\mathbf{U}^{(2)}$ has full column rank, where the latter is constructed from $(\mathbf{A}^{(1)}, \mathbf{A}^{(1)})$. As observed in [35], the symmetry introduces more type-(I) redundant rows in $\mathbf{U}^{(2)}$ than are identified in section 5.1. The following expression for the number of type-(I) nonredundant rows is conjectured by [35]:

$$\frac{I_1(I_1-1)}{4}\left(\frac{I_1(I_1-1)}{2}+1\right)-\binom{I_1}{4}, \tag{6.1}$$

where the last term only appears if $I_1 \geq 4$. In future research, we would like to identify redundant rows of $\mathbf{U}^{(n-1)}$ for nth order decompositions with various forms of symmetry (i.e., with some of $\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(n-1)}$ identical and $\mathbf{A}^{(n)}$ of rank R).

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